

Shifted Symplectic structures on derived stacks

/ k char $k = 0$

§ 1 - What is a stack?

Question: If G is an algebraic group, $G \curvearrowright X$ scheme,
does $[X/G]$ exist as a scheme?

A. No. \leftarrow integral.

Ex: $G = G_m$, $X = A^1$

Topologically A^1/G_m has 2 points $0 \in A^1/G_m$ closed

$G_m = 1 \in A^1/G_m$ open

$$= \{0, 1\} \quad \text{w. } \overline{\{1\}} = \{0, 1\}$$

This space has a scheme structure: $\text{Spec}(k[x]/x^2)$

But this is not the right object:

Global functions on $[A^1/G_m]$: $k[x]^{G_m} \cong k$.

\Rightarrow It doesn't exist as a scheme.

Functor of points: $\mathcal{C}\text{Ring} \rightarrow \text{Set}$

$$A \mapsto A^1(A)/G_m(A) = A/A^\times$$

Doesn't satisfy Zariski descent!

If it were rep. by a scheme it would!

We have to look at:

$$e\text{ Ring} \rightarrow \text{Groupoid}$$

$$F: A \mapsto \left. \begin{array}{l} \text{Ob: } x \in A \\ \text{Map: } x \xrightarrow{g \in A^*} y, \text{ s.t. } y = g \cdot x \end{array} \right\}$$

satisfies descent! To be more precise:

If U_i covers $U = \text{Spec } A$

$$\text{Then: } F(U) \simeq \left\{ \Delta_i \in F(U_i) + g_{ij}: \Delta_i|_{U_{ij}} \rightarrow \Delta_j|_{U_{ij}} \right\}$$

In fact, $F(A) = \left\{ \mathcal{L} \text{ line bundle on } \text{Spec } A + \{ f \in \Gamma(A, \mathcal{L}) \} \right\}$.

Def (Naive): A 1-stack is the data of a functor:

$$e\text{ Ring} \rightarrow \text{Groupoid}$$

satisfying descent.

Ex (Moduli stack)

$$A \mapsto \{ \text{v.l. on } \text{Spec } A \} / \simeq \quad \text{not descent.}$$

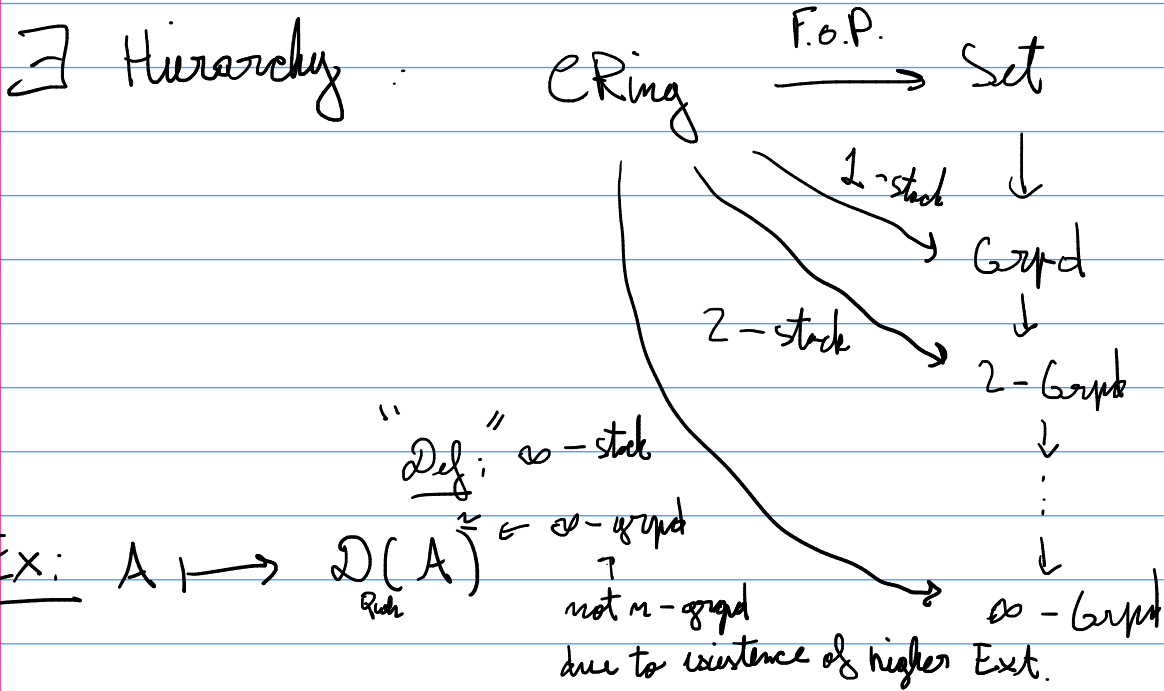
$$A \mapsto \text{Groupoid of v.l.} \quad \checkmark$$

! Higher stacks.

Ex: A gerbe on X is the data of a locally free sheaf of Groupoids on X .

$A \mapsto \{ \text{gerbes on } A \}$ \leftarrow group.
 doesn't descend.

$\{ \text{gerbes on } A \}^2 \leftarrow 2\text{-group}$.
 We need a def of 2-stacks!



§2. Donaldson-Thomas invariants

Goal: Count coherent sheaves on CY 3-folds.

(X, \mathcal{O}_X) CY if sm. proj. prop.
 $K_X \cong \mathcal{O}_X$

We study the stack:

$$M_X : A \mapsto \mathcal{D}(\text{Coh}_{X \times_{\text{Spec } A} \mathbb{A}^1}) \cong \text{is a stack}$$

It is the "moduli stack of coh. sh."

Deformation theory: $\mathcal{F} \in M_X$

1st order deformations of \mathcal{F}

are classified by: $\text{Ext}^1(\mathcal{F}, \mathcal{F})$.

$$+ \text{ob} \in \text{Ext}^2(\mathcal{F}, \mathcal{F})$$

We can rewrite it the following way:

$$\mathcal{D}(h) \ni \Pi_{M_x, \mathcal{F}} \cong \mathbb{R}\text{Hom}(\mathcal{F}, \mathcal{F})[1].$$

⚠ In fact, this is true if M_x is a derived stack.

Prop: If X (derived) stack, $\exists \mathbb{L}_X \in \mathcal{D}(X)$
(generalization of Ω_X^1)

$$\begin{array}{ccc} X' & \xrightarrow{\pi} & X \\ \downarrow \cong & \searrow & \downarrow \\ S' & \longrightarrow & S \end{array} \quad \text{then } \pi^* \mathbb{L}_{X/S} \cong \mathbb{L}_{X'/S'}$$

+ we have a fiber seq. $g: X \rightarrow Y$
triangle

$$g^* \mathbb{L}_Y \rightarrow \mathbb{L}_X \rightarrow \mathbb{L}_{X/Y}$$

$$\Pi_{M_x, \mathcal{F}} = \mathbb{R}\text{Hom}(\mathcal{F}, \mathcal{F})[1]$$

$$\mathbb{R}\Gamma(X, \mathcal{F} \otimes \mathcal{F}^\vee)[1]$$

$\mathbb{R} \leftarrow$ Serre Duality

$$\mathbb{R}\Gamma(X, \mathcal{F} \otimes \mathcal{F}^\vee \otimes K_X)[-2] \cong \Gamma(X, \mathcal{F} \otimes \mathcal{F}^\vee)[-2]$$

$$\cong \Pi_{M_x, \mathcal{F}}^\vee[-1] = \mathbb{L}_{M_x/\mathcal{F}}[-1]$$

\exists a symplectic form on M_x s.t. $\Pi_{M_x, \mathcal{F}} \cong \mathbb{L}_{M_x, \mathcal{F}}[-1]$
comes from it.

(Darboux Lemma for shifted symplectic str.)

$\leadsto M_x$ is locally $(\text{cut}(U, \mathcal{L}), \mathcal{L}: U \rightarrow \mathbb{A}^1)$
 \uparrow
 smooth scheme.

\exists Milnor number.

$\# M_x :=$ gluing of these Milnor numbers.

Def: An m -shifted symplectic on Y derived stack.

if the data of $\omega_0 \in \Lambda^2 \mathbb{L}_Y[m]$, s.t.

- $\Theta_{\omega_0}: \Pi_Y \xrightarrow{\text{q.l.}} \mathbb{L}_Y[m]. \quad \mathbb{L}_Y = (\mathbb{L}_Y^i, d)$
 - $d_{dR} \omega_0 \simeq 0$
- $d: \mathbb{L}_Y^i \rightarrow \mathbb{L}_Y^{i+1}$

data: $\omega_1 \in \Lambda^2 \mathbb{L}_Y[m-1]$

s.t. $d_{dR} \omega_0 = d\omega_1$

Symp (Y, m)

$:=$ ∞ -group of m -shifted symplectic forms.

\dots data $\omega_i \in \Lambda^{2+i} \mathbb{L}_Y[m-i]$

s.t. $d_{dR} \omega_i = d\omega_{i+1}$

Ex: $\mathcal{B}G = [*/G]$

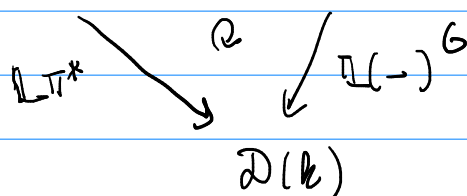
$\mathbb{L}G \hookrightarrow GL_m$

makes sense if G is algebraic.

In particular, $G \hookrightarrow GL_m$.

Prop: $\text{Symp}(\mathcal{B}G, m) = \begin{cases} \emptyset & \text{if } m \neq 2 \\ \text{Sym}^2(\mathbb{L}^V)^G, \text{nd.} & \text{if } m = 2. \end{cases}$

Pr (Quich). $\mathcal{D}(\mathcal{B}G) \simeq \mathcal{D}(\text{Rep}(G)) \quad * \xrightarrow{\pi} \mathcal{B}G$



Fact: $\mathbb{L}_{\mathbb{B}G} \cong \mathfrak{g}^{\vee}[1]$ \cong G coadj. action.

Cartan d^0 :

0	0	\mathfrak{g}^{\vee}	0	...
-1	0	1	2	

DR $(\mathbb{B}G)$:

			0	0	...
2	0	0	$\omega \in \text{Sym}^2(\mathfrak{g}^{\vee})^6$	0	...
1	0	$(\mathfrak{g}^{\vee})^6$	0	0	...
0	k^6	0	0	0	...
$w/\text{coh.}$	0	1	2	3	...

$\begin{matrix} \uparrow d_{dR} \\ \rightarrow 0 \xrightarrow{d} 0 \rightarrow \text{Sym}^3(\mathfrak{g}^{\vee})^6 \end{matrix}$

$\Rightarrow \text{Sym}(\mathbb{B}G, 2) \cong \text{Sym}^2(\mathfrak{g}^{\vee})^6, \text{nd.}$

$\text{Tr} \in \mathfrak{gl}_n.$

□.