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Shifted Symplectic structures on derived stacks

$/k \quad \mathrm{char}\, k = 0$

§1- What is a stack?

Question: If G is an algebraic group, $G \curvearrowright X$ scheme, does $[X/G]$ exists as a scheme? ← categorical.

A. No.

Ex: $G = \mathbb{G}_m$, $X = \mathbb{A}^1$

Topologically $\mathbb{A}^1/\mathbb{G}_m$ has 2 points $\overset{\mathbb{G}_m}{\curvearrowleft}\, 0 \in \mathbb{A}^1/\mathbb{G}_m$ closed

$\mathbb{G}_m/\mathbb{G}_m = 1 \in \mathbb{A}^1/\mathbb{G}_m$ open

$= \{0, 1\}$ w. $\overline{\{1\}} = \{0,1\}$

This space has a scheme structure: $\mathrm{Spec}\left(k[x]/x^2\right)$

But this is not the right object:

Global functions on $[\mathbb{A}^1/\mathbb{G}_m]$: $k[x]^{\mathbb{G}_m} \simeq k$.

$\Rightarrow$ It doesn't exist as a scheme.

Functor of points: $\mathrm{cRing} \longrightarrow \mathrm{Set}$

$A \longmapsto \mathbb{A}^1(A)/\mathbb{G}_m(A) = A/A^\times$

Doesn't satisfy Zariski descent!

If it were rep. by a scheme it would!

We have to look at:

$\mathcal{C}\mathrm{Ring} \longrightarrow \mathrm{Grpd}$

$$F : A \longmapsto \left\{ \begin{array}{l} \mathrm{Ob} : x \in A \\ \mathrm{Map} : x \xrightarrow{g \in A^*} y \ , \text{ s.t. } y = g \cdot x \end{array} \right\}$$

satisfies descent! To be more precise:

If U_i covers $U = \operatorname{Spec} A$

Then: $F(U) \simeq \{ s_i \in F(U_i) + g_{ij} : s_i|_{U_{ij}} \to s_j|_{U_{ij}} \}$

In fact, $F(A) = \{ \mathcal{L}, \text{ line bundle on } \operatorname{Spec} A \ +$
$f \in \Gamma(A, \mathcal{L}) \}$.

Def (Naive): A 1-stack is the data of a functor:

$\mathcal{C}\mathrm{Ring} \longrightarrow \mathrm{Grpd}$

satisfying descent.

Ex (Moduli stack)

$A \longmapsto \{ \text{v.b. on } \operatorname{Spec} A \} / \simeq$ not descent.

$A \longmapsto \mathrm{Grpd} \text{ of v.b.}$ ✓.

⚠ Higher stacks.

Ex: A gerbe on X is the data of a locally free sheaf of Groupoids on X.

$$A \longmapsto \{\text{gerbes on } A\} \leftarrow \text{grpd.}$$ doesn't sat. descent.

$$\cap$$

$$\{\text{gerbes on } A\}^{2} \leftarrow 2\text{-grpd.}$$

We need a def of 2-stacks!

$\exists$ Hierarchy:

$$\begin{tikzcd} \mathrm{CRing} \arrow[r, "\text{F.o.P.}"] \arrow[dr, bend left, "1\text{-stack}"] \arrow[ddr, bend left, "2\text{-stack}"] \arrow[ddddr, bend right] & \mathrm{Set} \arrow[d] \\ & \mathrm{Grpd} \arrow[d] \\ & 2\text{-Grpd} \arrow[d] \\ & \vdots \arrow[d] \\ & \infty\text{-Grpd} \end{tikzcd}$$

"Def:" ∞-stack

Ex: $A \longmapsto \mathcal{D}_{\mathrm{Qcoh}}(A)^{\simeq} \leftarrow \infty$-grpd

not n-grpd due to existence of higher Ext.

§2. Donaldson-Thomas invariants

Goal: Count coherent sheaves on CY 3-folds.

(X/k CY if . sm. proj. prop. $K_X \simeq \mathcal{O}_X$)

We study the stack:

$$\mathcal{M}_X : A \longmapsto \mathcal{D}(\mathrm{Coh}_{X \times \mathrm{Spec} A / A})^{\simeq}$$ is a stack

It is the "moduli stack of coh. sh."

Deformation theory: $\mathcal{F} \in \mathcal{M}_X$

1st order deformations of $\mathcal{F}$

are classified by : $\mathrm{Ext}^1(\mathcal{F},\mathcal{F})$.

+ obs $\in \mathrm{Ext}^2(\mathcal{F},\mathcal{F})$

We can rewrite it the following way:

$$\mathcal{D}(k) \ni \mathbb{T}_{\mathcal{M}_X,\mathcal{F}} \simeq \mathbb{R}\mathrm{Hom}(\mathcal{F},\mathcal{F})[1].$$

⚠ In fact, this is true if $\mathcal{M}_X$ is a derived stack.

Prop: If X (derived) stack, $\exists\, \mathbb{L}_X \in \mathcal{D}(X)$

(generalization of Ω^1_X)

$$\begin{tikzcd} X' \arrow[r, "\pi"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

then $\pi^* \mathbb{L}_{X/S} \simeq \mathbb{L}_{X'/S'}$

+ We have a fiber seq. $f: X \to Y$

triangle

$$f^*\mathbb{L}_Y \to \mathbb{L}_X \to \mathbb{L}_{X/Y}.$$

$$\mathbb{T}_{\mathcal{M}_X,\mathcal{F}} = \mathbb{R}\mathrm{Hom}(\mathcal{F},\mathcal{F})[1]$$
$$\parallel$$
$$\mathbb{R}\Gamma(X, \mathcal{F}\otimes\mathcal{F}^\vee)[1]$$
$$\wr\!\wr \leftarrow \text{Serre Duality}$$
$$\mathbb{R}\Gamma(X, \mathcal{F}\otimes\mathcal{F}^\vee\otimes K_X)[-2] \simeq \Gamma(X,\mathcal{F}\otimes\mathcal{F}^\vee)[-2]$$
$$\simeq \mathbb{T}^\vee_{\mathcal{M}_X,\mathcal{F}}[-1] = \mathbb{L}_{\mathcal{M}_X/\mathcal{F}}[-1]$$

$\exists$ a symplectic form on $\mathcal{M}_X$ s.t. $\mathbb{T}_{\mathcal{M}_X,\mathcal{F}} \simeq \mathbb{L}_{\mathcal{M}_X,\mathcal{F}}[-1]$ comes from it.

(Darboux Lemma for shifted symp. str.)

$\rightsquigarrow M_X$ is locally $(\mathrm{Crit}(U, f).\ f: U \to \mathbb{A}^1)$

smooth scheme.

$\exists$ Milnor number.

$\# M_X :=$ Gluing of these Milnor numbers.

Def: An n-shifted symplectic on Y derived stack.

it the data of $\omega_0 \in \Lambda^2 \mathbb{L}_Y[n]$, s.t.

- $\theta_{\omega_0}: \mathbb{T}_Y \xrightarrow[q.i.]{} \mathbb{L}_Y[n]$. $\mathbb{L}_Y = (\mathbb{L}_Y^i, d)$ $d_i: \mathbb{L}_Y^i \to \mathbb{L}_Y^{i+1}$

- $d_{dR}\omega_0 \simeq 0$

data: $\omega_1 \in \Lambda^3 \mathbb{L}_Y[n-1]$

s.t. $d_{dR}\omega_0 = d\omega_1$

$\mathrm{Symp}(Y, n)$ $:= \infty$-grpd of n-shifted symp. forms.

... data $\omega_i \in \Lambda^{2+i} \mathbb{L}_Y[n-i]$

s.t. $d_{dR}\omega_i = d\omega_{i+1}$.

Ex: $\mathcal{B}G = \left[{*}/{G}\right]$ makes sense if G is algebraic.

If $G \hookrightarrow GL_n$ In particular, $G \hookrightarrow GL_n$.

Prop: $\mathrm{Symp}(\mathcal{B}G, n) = \begin{cases} \emptyset & \text{if } n \neq 2 \\ \mathrm{Sym}^2(\mathfrak{g}^\vee)^{G,\,\mathrm{nd}.} & \text{if } n = 2. \end{cases}$

Pf (Quick). $\mathcal{D}(\mathcal{B}G) \simeq \mathcal{D}(\mathrm{Rep}(G))$ $\quad * \xrightarrow{\pi} \mathcal{B}G$

$$\begin{tikzcd} \mathcal{D}(\mathcal{B}G) \arrow[dr, "\mathbb{L}\pi^*"'] \arrow[rr, phantom, "\simeq"] & & \mathcal{D}(\mathrm{Rep}(G)) \arrow[dl, "\Gamma(-)^G"] \\ & \mathcal{D}(k) \arrow[u, phantom, "\circlearrowright"] & \end{tikzcd}$$

Fact: $\mathbb{L}_{BG} \simeq \mathfrak{g}^\vee[1] \circlearrowleft G$ coadj. action.

$$\begin{array}{lcccccc}
 & 0 & 0 & \mathfrak{g}^\vee & 0 & \cdots \\
\text{coho } d^\circ: & -1 & 0 & 1 & 2 &
\end{array}$$

$$\begin{array}{c}
0 \\
\to 0 \xrightarrow{d} 0 \longrightarrow \mathrm{Sym}^3(\mathfrak{g}^\vee)^G \\
\uparrow d_{dR}
\end{array}$$

$$\begin{array}{llcccc}
 & \vdots & & & & \\
DR(BG): & 2 & 0 & 0 & \omega \in \mathrm{Sym}^2(\mathfrak{g}^\vee)^G & \\
 & 1 & 0 & (\mathfrak{g}^\vee)^G & 0 & \\
 & 0 & k^G & 0 & 0 & \cdots \\
 & w/\text{coh.} & 0 & 1 & 2 & 3 \;\cdots
\end{array}$$

$\Rightarrow \mathrm{Symp}(BG, 2) \simeq \mathrm{Sym}^2(\mathfrak{g}^\vee)^{G,\text{nd}}.$

$\qquad\qquad\qquad\qquad\qquad \ni$ Tr in GL_n. $\square$