INTRODUCTION TO TORIC VARIETIES

I · Motivation

$(\mathbb{C}^\times)^n$ torus

$(\mathbb{C}^\times)^n \times \mathbb{P}^n_{\mathbb{C}} \longrightarrow \mathbb{P}^n_{\mathbb{C}}$

$((z_1,\dots,z_n),[x_0:\dots:x_n]) \longmapsto [x_0 : z_1x_1 : \dots : z_nx_n]$

Restrict it to $(\mathbb{C}^\times)^n \subseteq \mathbb{P}^n_{\mathbb{C}}$
$[1:z_1:\dots:z_n]$

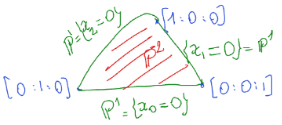
↳ translation action of $(\mathbb{C}^\times)^n$ on itself

Toric variety $\simeq$ alg var$/\mathbb{C}$ $\underset{\text{dense}}{\supseteq}$ torus T $\circlearrowleft T$ s.t $T \curvearrowright T$ translation action

$\underline{n=2}$: $T = (\mathbb{C}^\times)^2 \subseteq \mathbb{P}^2_{\mathbb{C}} \ni [x_0:x_1:x_2]$

$\{x_i = 0\} = \mathbb{P}^1_{\mathbb{C}} \circlearrowleft T$

$\{x_i = x_j = 0\} = \{[1:0:0]\}$ invariant



Polyhedral shape $\longleftrightarrow$ Combinatorics of the action

II · Lattices, cones & fans

$(\mathbb{Z}^n \simeq) M$ free ab gp of rk n

$M_{\mathbb{R}} = M \otimes \mathbb{R} \ (\simeq \mathbb{R}^n)$

$N = \mathrm{Hom}(M,\mathbb{Z})$, $N_{\mathbb{R}}$

Cone : Convex, polyhedral, rational, closed

$t \in \mathbb{R}_+$ $\quad 0 \leq t \leq 1$, $v,w \in \sigma$, $(1-t)v + tw \in \sigma$

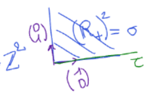
gen by a finite set $\sigma = \mathbb{R}_+ v_1 + \mathbb{R}_+ v_2 + \dots + \mathbb{R}_+ v_k$, $\in N$

$\sigma \subseteq N_{\mathbb{R}}$ $\quad v \in \sigma \Rightarrow tv \in \sigma$

$M_{\mathbb{R}} = (N_{\mathbb{R}})^* \supseteq \sigma^\perp = \{ l \in M_{\mathbb{R}} \mid l(v) = 0 \ \forall v \in \sigma\}$

$\cap$ $\quad \sigma^\vee = \{ l \mid \forall v \in \sigma, l(v) \geq 0\}$

Ex: $N = \mathbb{Z}^2$, M



$\sigma^\vee = (\mathbb{R}_+)^2$

$\tau^\vee = \mathbb{R}_+ \times \mathbb{R} \supseteq \tau^\perp = \{0\} \times \mathbb{R}$

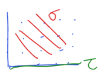
both are rational (Farkas lemma)

$\sigma \subseteq N_{\mathbb{R}}$, $\tau \subseteq \sigma$ subcone

face $\tau < \sigma$ if $\forall v,w \in \sigma$ $\quad v + w \in \tau \Rightarrow v,w \in \tau$

$\left(\Leftrightarrow \quad \tau = \sigma \cap u^\perp \quad \exists u \in M_{\mathbb{R}}, \ u \in \sigma^\vee \right)$

Rat cone $\sigma \subseteq N_{\mathbb{R}}$, $\sigma \cap N$ is comm. monoid $\quad$ $\mathbb{C}$-alg



$(\mathbb{R}_+)^2 \cap N = \mathbb{N}^2 = \sigma \cap N$

$\tau \cap N = \mathbb{N}$

Gordan's lemma; f.g monoid

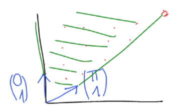
Affine toric variety $\sigma \subseteq N_{\mathbb{R}}$

$M \supseteq \sigma^\vee \cap M$, $\mathbb{C}[\sigma^\vee \cap M]$ f.g $\mathbb{C}$-alg

$U_\sigma = \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]$ $\quad (S_\sigma = \sigma^\vee \cap M)$

$S_\sigma = \langle v_1, \dots, v_k \rangle \rightsquigarrow \mathbb{C}[S_\sigma] = \mathbb{C}[X_1,\dots,X_k] / \left(\prod X_i^{\lambda_i} - \prod X_j^{\mu_j}\right)$

$\sum \underset{\geq 0}{\lambda_i} v_i = \sum \underset{\geq 0}{\mu_j} v_j$, $\in \mathbb{Z}$



$\tau < \sigma$ $\quad U_\tau \subseteq U_\sigma$ principal open

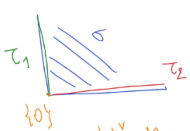
Ex:

$M \cap \sigma^\vee = \mathbb{N}^2 \longrightarrow U_\sigma = \mathbb{C}^2 \supseteq \{y \neq 0\}$

$M \cap \tau^\vee = \mathbb{N} \times \mathbb{Z} \longrightarrow U_\tau = \mathbb{C} \times \mathbb{C}^\times$

Fan Σ set of rat'l cones $\subseteq N_{\mathbb{R}}$

s.t $\begin{cases} \tau < \sigma \in \Sigma \Rightarrow \tau \in \Sigma \\ \sigma_1, \sigma_2 \in \Sigma \Rightarrow \sigma_1 \cap \sigma_2 \in \Sigma \end{cases}$



$\{0\}^\vee = M_{\mathbb{R}}$

$M \simeq \mathbb{Z}^2$

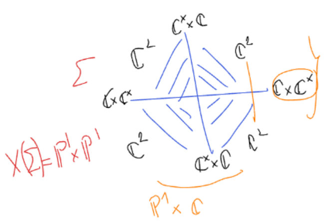
$U_\sigma = \mathbb{C}^2$

$U_{\tau_1} = \mathbb{C}^\times \times \mathbb{C}$

$U_{\tau_2} = \mathbb{C} \times \mathbb{C}^\times$

$U_{\{0\}} = \mathbb{C}^\times \times \mathbb{C}^\times$

$U_{\tau_1} \cap U_{\tau_2} = U_{\tau_1 \cap \tau_2}$



Fan Σ

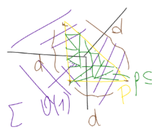
$\downarrow$

$X(\Sigma) = \operatorname{colim}_{\sigma \in \Sigma} U_\sigma$

III - Toric var. dictionary

$X(\Sigma)$	Fan Σ
$U_\tau \subseteq U_\sigma$	$\tau < \sigma \in \Sigma$
$V_\sigma(\tau) = V(\tau)$ clsd in $X(\Sigma)$; $\cap$ fixed k-coord.: 0; $U_\sigma = \mathbb{C}^l \times (\mathbb{C}^\times)^{n-l}$	$\tau = (\mathbb{R}_+)^k$, $\wedge$, $\sigma = (\mathbb{R}_+)^l$
$U_\sigma \supseteq V_\sigma(\tau) \to V(\tau) \circlearrowleft T$; 1 eqn; hyp of X	τ ray
$T \subseteq X \curvearrowleft$ $N \otimes \mathbb{C}^\times = T$	$N = \{0\}^\vee \cap N$, $\{0\} \in \Sigma$
$\dim X$	$\mathrm{rk}\, N$

IV - Divisors



$X(\Sigma) = \mathbb{P}^2 \circlearrowleft \mathcal{O}(d)$

$\bigcup X_p$ reducible var

$\pi^{-1}(0) = P$

$\mathcal{X} \xrightarrow{\pi} \mathbb{C}$ toric var

$\pi^{-1}(t) = \mathbb{P}^2 \supseteq C_t$

$\mathcal{C}$ nice

$\mathcal{C}_{\{0\}} \subseteq \pi^{-1}(0)$

$\mathcal{C}_p \subseteq X_p$ "line"

$\mathcal{C}_{\{0\}} \subseteq$

$\mathcal{C}_p$