TRIVIALIZING THE G-ACTION AND THE VECTOR BUNDLE

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Statement. Consider a finite étale Galois cover $\pi : X \to Y$ of schemes with group G, hence $Y \simeq X/G$. Let $\mathcal{F}$ be a locally free sheaf of finite rank on X. We are concerned with the question of whether one can find G-invariant open subschemes U_i covering X and trivializing $\mathcal{F}$. In this note we show that this is the case if and only if $\mathcal{F}$ is isomorphic to the pullback of a locally free sheaf on the quotient X/G[1]

Proof. If $\mathcal{F} \simeq \pi^*\mathcal{M}$ for some locally free sheaf $\mathcal{M}$ on Y, then one can pick V_i a cover of Y trivializing $\mathcal{M}$ and set $U_i = V_i$. Conversely, assume such a cover $\{U_i \to X\}$ exists. We will define a descent datum $(\mathcal{F}, \phi)$ associated to the fpqc cover $\{X \to Y\}$. To this end, notice that for every $g \in G$ there is a morphism of locally free sheaves

$$\phi_g : \mathcal{F} \to g^*\mathcal{F}$$

sending a local section s to the precomposition $s \circ g^{-1}$. If $h \in G$ is another group element, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\quad\phi_g\quad} & g^*\mathcal{F} \\ & {\scriptstyle \phi_{hg}} \searrow \quad \swarrow {\scriptstyle g^*\phi_h} & \\ & (hg)^*\mathcal{F} & \end{array}$$

If we can show that each ϕ_g is an isomorphism, then the collection of these morphisms will define a descent datum (the cocycle condition being ensured by the commutative triangle above).

Consider some U_i, fix an isomorphism $\mathcal{F}_{|U_i} \simeq \mathcal{O}_{U_i}^n$. Over U_i the morphism ϕ_g decomposes as

$$\mathcal{F}_{|U_i} \simeq \mathcal{O}_{U_i}^n \xrightarrow{\sim} g^*\mathcal{O}_{U_i}^n \simeq g^*\mathcal{F}_{|U_i}$$

which is an isomorphism. As the U_i cover X, we find that ϕ_g is an isomorphism globally. It remains to show that the collection of all ϕ_g can be packaged to form a descent datum. To this end, denote by p_1 and p_2 the two projection maps $X \times_Y X \to X$, by q_1, q_2 and q_3 the three projections $X \times_Y X \times_Y X$, and by pr_{12}, pr_{13} and pr_{23} the three projections $X \times_Y X \times_Y X \to X \times_Y X$. Recall that a descent datum is an isomorphism

$$\phi : p_1^*\mathcal{F} \to p_2^*\mathcal{F}$$

satisfying the cocycle condition

$$\begin{array}{ccc} q_1^*\mathcal{F} & \xrightarrow{\quad\mathrm{pr}_{12}^*\phi\quad} & q_2^*\mathcal{F} \\ & {\scriptstyle \mathrm{pr}_{13}^*\phi} \searrow \quad \swarrow {\scriptstyle pr_{23}^*\phi} & \\ & q_3^*\mathcal{F} & \end{array}$$

Since π is étale there are two identifications $X \times_Y X \simeq \coprod_{g \in G} X_g$ (where the X_g are copies of X):

$$\begin{array}{ccc} \coprod_{g \in G} X_g \xrightarrow{f_1} & X \times_Y X \xrightarrow{p_1} & X \\ x \in X_h \mapsto & (x, hx) \mapsto & x \end{array}$$

and

$$\begin{array}{ccc} \coprod_{g \in G} X_g \xrightarrow{f_2} & X \times_Y X \xrightarrow{p_2} & X \\ x \in X_h \mapsto & (hx, x) \mapsto & x. \end{array}$$

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[1]This question was asked to me by Matteo Verni and discussed between him, Tiago Landim, Drimik Roy and myself. I thank them for explaining the argument to me.

These two identifications compose to an automorphism of $\coprod_{g\in G} X_g$:

$$\begin{array}{ccccc} \coprod_{g\in G} X_g & \xrightarrow{f_1} & X\times_Y X & \xrightarrow{f_2^{-1}} & \coprod_{g\in G} X_g \\ x\in X_h & \mapsto & (x,hx) & \longmapsto & hx\in X_{h^{-1}}. \end{array}$$

Let $r = f_2^{-1}\circ f_1$. We see that $r_{|X_h} : X_h \to X_{h^{-1}}$ is the action of h on X (in particular, r is an involution). To summarize:

- the pullback $(p_1\circ f_1)^*\mathcal{F}$ is the sheaf $\mathcal{F}$ on every copy X_h in $\coprod_{g\in G} X_g$;
- similarly for $(p_2\circ f_2)^*\mathcal{F}$;
- the pullback $(p_2\circ f_1)^*\mathcal{F} = (p_2\circ f_2\circ r)^*\mathcal{F}$ is, for every $h\in G$, the sheaf $h^*\mathcal{F}$ on X_h.

Therefore, an isomorphism $\phi : p_1^*\mathcal{F}\xrightarrow{\sim} p_2^*\mathcal{F}$ amounts to an isomorphism

$$(p_1\circ f_1)^*\mathcal{F}\xrightarrow{\sim}(p_2\circ f_1)^*\mathcal{F}$$

which, according to the description above, amounts to a collection of isomorphisms

$$\forall h\in G,\ \mathcal{F}\xrightarrow{\sim} h^*\mathcal{F}.$$

Similarly, we can show that the cocycle condition is tantamount to the commutative diagram

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & g^*\mathcal{F} \\ & \searrow \quad \swarrow & \\ & (hg)^*\mathcal{F} & \end{array}$$

This shows that the ϕ_g defined at the beginning do indeed constitute a descent datum. As fpqc descent is effective for quasi-coherent sheaves on schemes (see [Sta26, Tag 023T]), this shows that $\mathcal{F}\simeq\pi^*\mathcal{M}$ for some locally free sheaf $\mathcal{M}$ on Y. Of course, étale descent would suffice.

References

[Sta26] T. Stacks project authors – "The stacks project", https://stacks.math.columbia.edu, 2026.

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