THE RELATIVE DE RHAM THEOREM

FRANÇOIS GATINE

1. Introduction

Let $\pi : X \to Y$ be a proper submersion of real smooth manifolds. We denote by $\mathcal{C}_X$ and $\mathcal{C}_Y$ the sheaves of real-valued smooth functions on the respective spaces, as well as Ω^k_X and Ω^k_Y the sheaves of real-valued k-differential forms. We also consider $\Omega^k_{X/Y} := \Omega^k_X/(\pi^*\Omega^1_Y \wedge \Omega^{k-1}_X)$ the sheaf of relative differential forms. The goal of this note is to establish canonical isomorphism of $\mathcal{C}_Y$-modules:

$$R\pi_*(\Omega^\bullet_{X/Y}) \simeq R\pi_*\underline{\mathbb{R}}_X \otimes_{\underline{\mathbb{R}}_Y} \mathcal{C}_Y.$$

which was stated in [Voi03]. Taking k-th cohomology and looking at the fiber above $y \in Y$ yields the de Rham theorem for X_y:

$$H^k_{\mathrm{dR}}(X_y) \simeq H^k_{\mathrm{sing}}(X_y, \mathbb{R}).$$

To see that the fiber on the left-hand side is indeed the de Rham cohomology of X_y, simply apply the proper base-change theorem to the pullback diagram

$$\begin{array}{ccc} X_y & \longrightarrow & X \\ \big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ y & \longrightarrow & Y \end{array}.$$

The argument below was found by Matteo Verni and Tangi Pasquer. I would like to thank them for sharing their proof.

2. Proof

We show that both sides are canonically isomorphic to $R^k\pi_*(\pi^{-1}\mathcal{C}_Y)$.

2.1. We claim that the sheaf complex $\Omega^\bullet_{X/Y}$ is an acyclic resolution of $\pi^{-1}\mathcal{C}_Y$, which implies the equality

$$R\pi_*(\pi^{-1}\mathcal{C}_Y) = R\pi_*(\Omega^\bullet_{X/Y}).$$

First, notice that each sheaf $\Omega^k_{X/Y}$ is a $\mathcal{C}_X$-module, which makes it acyclic for π_*. The fact that they define a resolution follows from the Poincaré lemma[1]. Now it suffices to see that the kernel of $\mathcal{C}_X \xrightarrow{d_{X/Y}} \Omega^1_{X/Y}$ is $\pi^{-1}\mathcal{C}_Y$. For this, we may work locally on X and assume $Y = \mathbb{R}^m$, $X = \mathbb{R}^n \times Y$. If $f \in \mathcal{C}_X$ is in the kernel then all of its partial derivatives along the horizontal components vanish: f is constant in the $\mathbb{R}^n$-component of its input, hence comes from a function on Y, as desired.

2.2. It remains to show the isomorphism $R\pi_*(\pi^{-1}\mathcal{C}_Y) = R\pi_*\underline{\mathbb{R}}_X \otimes_{\underline{\mathbb{R}}_Y} \mathcal{C}_Y$. The proof is a formal argument in derived functors formalism. Notice that we can write

$$\pi^{-1}\mathcal{C}_Y = \pi^{-1}\mathcal{C}_Y \otimes_{\underline{\mathbb{R}}_X} \underline{\mathbb{R}}_X = \pi^{-1}\mathcal{C}_Y \otimes^{\mathrm{L}}_{\underline{\mathbb{R}}_X} \underline{\mathbb{R}}_X$$

where $\otimes^{\mathrm{L}}$ denotes the derived tensor product. Note that we have done essentially nothing profound yet, as the functor $- \otimes_{\underline{\mathbb{R}}_X} \underline{\mathbb{R}}_X$ is quite clearly the identity (hence so is $- \otimes^{\mathrm{L}}_{\underline{\mathbb{R}}_X} \underline{\mathbb{R}}_X$). Now we may apply the derived projection formula, which, in the context of proper maps[2] yields

$$R\pi_*(\pi^{-1}\mathcal{C}_Y) = R\pi_*(\pi^{-1}\mathcal{C}_Y \otimes^{\mathrm{L}}_{\underline{\mathbb{R}}_X} \underline{\mathbb{R}}_X) = \mathcal{C}_Y \otimes^{\mathrm{L}}_{\underline{\mathbb{R}}_Y} R\pi_*\underline{\mathbb{R}}_X.$$

Date: February 4, 2026.

[1] The Poincaré lemma holds in the relative setting: note that we can work locally, and thus assume $Y = \mathbb{R}^m$, $X = \mathbb{R}^n \times Y$. The sheaves of differential forms split nicely, and we are (almost) reduced to the standard Poincaré lemma.

[2] This means that the $\pi_!$ and $\pi^!$ in the six-functor formalism reduce to the standard π_* and π^{-1}.

Notice now that the functor $\mathcal{C}_Y \otimes_{\mathbb{R}_Y} -$ is exact[3], so the derived tensor product is a regular tensor product. This establishes the isomorphism and concludes the proof.

References

[Voi03] C. Voisin – *Hodge theory and complex algebraic geometry ii*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 2003.

Email address: francois.gatine@imj-prg.fr

[3] Easily seen by looking at the stalks where we are tensoring real vector spaces.