ACYCLICITY OF CERTAIN LOCAL SYSTEMS ON THE UNITARY BUNDLE OF HYPERBOLIC SURFACES

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1. Introduction

This note contains an alternative proof of Lemma 1.2, (2), from [HT26].

Proposition 1. *Let Σ be a hyperbolic surface of genus $g \geq 2$, denote by M its unitary bundle with respect to its metric. Consider a local system L over M corresponding to a representation*

$$\rho : \pi_1(M) \to \mathrm{GL}_r(\mathbb{C}).$$

Assume that ρ is irreducible, and that it does not factor through $\pi_1(\Sigma)$ via the functorially induced map $\pi_1(M) \to \pi_1(\Sigma)$. Then the sheaf L is acyclic

$$H^i(M, L) = 0 \qquad \forall i \geq 0.$$

The proof also covers the algebraic counterpart of the statement.

Proposition 2. *Let k be an algebraically closed field of characteristic zero, ℓ be a prime number and Σ be a hyperbolic curve of genus $g \geq 2$, denote by M the complement of the zero section in its tangent bundle. Consider an étale local system L over M corresponding to a continuous representation*

$$\rho : \pi_1(M) \to \mathrm{GL}_r(\overline{\mathbb{Q}_\ell}).$$

Assume that ρ is irreducible, and that it does not factor through $\pi_1(\Sigma)$ via the functorially induced map $\pi_1^{\text{ét}}(M) \to \pi_1^{\text{ét}}(\Sigma)$. Then the sheaf L is acyclic

$$H^i_{\text{ét}}(M, L) = 0 \qquad \forall i \geq 0.$$

2. Proof of the proposition

We let F denote the circle S^1 (*resp.* the punctured line $\mathbb{A}^1 \setminus \{0\}$) in the differential setting (*resp.* in the algebraic setting). We will write indistinguishingly π_1 and H^i for the classical and étale fundamental groups and cohomology groups. Denote by W the underlying space for the representation ρ over the field $K = \mathbb{C}$ or $\overline{\mathbb{Q}_\ell}$.

2.1. Presentations of fundamental groups.

This paragraph follows [HT26]. Notice that we have a fibration

$$F \to M \xrightarrow{f} \Sigma. \tag{1}$$

From there it is possible to show that the fundamental groups have presentations

$$\pi_1(\Sigma) = \langle a_1, b_1, \dots, a_g, b_g \,|\, [a_1, b_1] \dots [a_g, b_g] \rangle,$$
$$\pi_1(M) = \langle a_1, b_1, \dots, a_g, b_g, c \,|\, [a_i, c], [b_i, c], [a_1, b_1] \dots [a_g, b_g] c^{2-2g} \rangle.$$

and the map $\pi_1(M) \to \pi_1(\Sigma)$ is the obvious one with kernel $\pi_1(F)$ generated by c. Thus we see that the assumption that ρ does not factor through $\pi_1(\Sigma)$ is equivalent to $\rho(c) \neq \mathrm{id}$. As c is in the center of $\pi_1(M)$ and ρ is irreducible, Schur's lemma states that $\rho(c) = \zeta \cdot \mathrm{id}$ for some $\zeta \in K \setminus \{1\}$. In particular:

Lemma 1. *The invariant subspace $W^{\pi_1(F)}$ and the quotient space $W/(\rho(c) - \mathrm{id})W$ are trivial.*

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2.2. **Spectral sequence.** We consider the Leray-Serre spectral sequence attached to the fibration 1:

$$E_2^{p,q} = H^q(\Sigma, R^p f_* L) \implies H^{p+q}(M, L).$$

By standard properties of cohomology with respect to dimension, $H^i(M, L)$ is automatically zero for $i > 3$, hence we focus on $i \in \{0, 1, 2, 3\}$. By the same argument, $E_2^{p,q}$ vanishes for $q > 2$, as well as $R^p f_* L$ for $p > 1$[1]. What is left of the E_2 page is the following:

	0	1
2	$H^2(\Sigma, f_* L)$	$H^2(\Sigma, R^1 f_* L)$
1	$H^1(\Sigma, f_* L)$	$H^1(\Sigma, R^1 f_* L)$
0	$H^0(\Sigma, f_* L)$	$H^0(\Sigma, R^1 f_* L)$

We claim that the sheaves $f_* L$ and $R^1 f_* L$ are zero, which implies that the E_2-page is zero and concludes the proof. Indeed, $R^i f_* L$ is the local system on Σ whose fiber above $b \in \Sigma$ identifies with $H^i(F, L_{|F})$:

- for $i = 0$, this is the space of global sections of $L_{|F}$ which identify with the invariant subset $W^{\pi_1(F)}$, and
- for $i = 1$, it coincides with $H^1(\pi_1(F), W) = W/(\rho(c) - \mathrm{id})W$.

Lemma 1 shows that these fibers are zero.

References

[HT26] T. Humbert & Z. Tao – "Twisted pollicott–ruelle resonances and zeta function at zero on surfaces", 2026.

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[1]Recall that the fibers of $R^p f_* L$ identify with $H^p(F, L_{|F})$.