About how computing log-crystalline cohomology of the rigid analytic line boils down to graph homology

I/ Short motivation from p-adic Hodge theory

Let X be a $\mathbb{Q}_p$-rigid analytic variety. When X is smooth and proper, its étale cohomology $H^i_{ét}(X_{\mathbb{C}_p}, \mathbb{Q}_p)$ are finite dimensional continuous representations of the absolute Galois group $G_{\mathbb{Q}_p}$, over $\mathbb{Q}_p$.

From 1980, J.M. Fontaine defined the notion of semistable representations. Roughly there exists $B_{st} \circlearrowleft G_{\mathbb{Q}_p}, \varphi, N$, filtration on B_{dR} and V is semistable iff

$$B_{st} \otimes_{\mathbb{Q}_p} V \simeq B_{st} \otimes_{\mathbb{Q}_p} D \text{ for some filtered } (\varphi, N)\text{-module}.$$

<u>Conj</u> (Fontaine) the above $H^i_{ét}(X_{\mathbb{C}_p}, \mathbb{Q}_p)$ are semistable.

But what plays the role of D ?

<u>Upshot</u>: log-crystalline cohomology of a semistable model, with its φ, N and filtration coming from comparison with de Rham cohomology.

<u>Def</u>: (rough) A $\mathbb{Q}_p$-rigid analytic variety X is an "admissible gluing" of

$$\mathrm{SpB}(\mathbb{Q}_p\langle X_1, \dots, X_d\rangle / I) = \{\text{multiplicative semi-norms} \leq \| \ \|\} \text{ with suitable topology}$$

and structure sheaf.

Here $\mathbb{Q}_p\langle X_1, \dots, X_d\rangle = \{ \sum_{\underline{\delta} \in \mathbb{N}^d} a_{\underline{\delta}} X^{\underline{\delta}} \mid a_{\underline{\delta}} \xrightarrow[\underline{\delta} \to +\infty]{} 0 \}$ with Banach norm $\| \ \|$ given by the supremum norm. I is a finitely generated ideal.

<u>Ex</u>: 1) $\mathbb{B}^{1,an}_{\mathbb{Q}_p} = \mathrm{SpB}(\mathbb{Q}_p\langle X\rangle)$ functions that converge on $|x| \leq 1$.

2) $\mathbb{A}^{1,an}_{\mathbb{Q}_p} = \mathrm{Colim}\ \mathrm{SpB}(\mathbb{Q}_p\langle X_n\rangle)$ with $X_{n+1} = \frac{X_n}{p}$, colimit of increasing radii balls.

3) $C^{1,an}_{\mathbb{Q}_p} = \mathrm{SpB}(\mathbb{Q}_p\langle X, Y\rangle / XY - p)$ functions that converge on the annulus $\frac{1}{p} \leq |x| \leq 1$

<u>Philosophy</u>: we glued basic pieces i.e. closed variety inside polydiscs as for algebraic varieties we glued closed varieties inside affine spaces.

Def: A model of X is a p-adic formal scheme $\mathfrak{X}/\mathrm{Spf}\,\mathbb{Z}_p$ st its generic fiber $\mathfrak{X}_\eta$ identifies to X.

It is called semistable if it is étale locally étale over an annulus
$$\mathrm{Spf}\left(\mathbb{Z}_p\langle X_1,\dots,X_d\rangle / X_1\cdots X_d - p\right).$$

Philosophy: a p-adic formal scheme is a p-complete scheme.

Th (Fontaine-Messing ~80, Faltings and Tsuji ~90 in general)
For X smooth proper with good reduction semistable model $\mathfrak{X}$
$$B_{st} \otimes_{\mathbb{Q}_p} H^i_{et}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p) \simeq B_{st} \otimes_{\mathbb{Z}_p} H^i_{logcrys}(\mathfrak{X}_s / \mathrm{Spf}\,\mathbb{Z}_p^\circ)$$

2/ Definitions and scholies

Def: A prelogscheme (resp. a logscheme) is a triple $(Z, \mathcal{M}, \alpha)$ where Z is a scheme, $\mathcal{M}$ is an étale sheaf of monoids and $\alpha: \mathcal{M} \to (\mathcal{O}_Z, \times)$ (resp. such that $\alpha: \alpha^{-1}(\mathcal{O}_Z^\times) \simeq \mathcal{O}_Z^\times$).
There is a right adjoint to prelog $\subset$ log.
There is notions of logsmooth, logétale, log de Rham complex that coincide with the usual when $\mathcal{M} = \mathcal{O}_Z^\times$.

Philosophy: autorize logarithmic poles on differential forms i.e. $\frac{dx}{x}$ for $x \in \mathrm{Im}(\mathcal{M})$

Ex: Take $Z = \mathrm{Spec}\left(\mathbb{F}_p[X,Y]/XY\right)$ with logstructure coming from $\underline{\mathbb{N}}^2 \to \mathcal{O}_Z$, $e_1 \mapsto X$, $e_2 \mapsto Y$

over $(\mathrm{Spec}\,\mathbb{F}_p)^\circ$ — $\mathbb{N} \to \mathbb{F}_p$, $1 \mapsto 0$, it is logsmooth.

Def (Berthelot ~70, Hyodo-Kato ~90): For a (fine) $\mathrm{Spec}(\mathbb{F}_p)^\circ$-logscheme Z, we have an analogue of the crystalline sites $(Z/\mathrm{Spec}(\mathbb{Z}/p^n)^\circ)_{logcrys}$ using similar logstructure on the canonical DP-structure on $p\,\mathbb{Z}/p^n\mathbb{Z}$. They have structure sheaves $\mathcal{O}_{Z/\mathbb{Z}/p^n, logcrys}$. Define
$$R\Gamma_{logcrys}(Z) := \varprojlim R\Gamma_{logcrys}(Y, \mathcal{O}_{Z/\mathbb{Z}/p^n, logcrys}) \quad \text{in } D(\mathbb{Z}_p).$$

It has Frobenius coming from
$$\begin{tikzcd} Z \arrow[r, "\varphi"] \arrow[d] & Z \arrow[d] \\ (\mathrm{Spf}\,\mathbb{Z}_p)^0 \arrow[r, "\varphi"'] & (\mathrm{Spf}\,\mathbb{Z}_p)^0 \end{tikzcd}$$
and monodromy which we ignore. under mild condition.

<u>Prop</u> : Suppose we have
$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d] & Z' \arrow[d, "g"] \\ \mathrm{Spec}\,\mathbb{F}_p^0 \arrow[r] & \mathrm{Spec}(\mathbb{Z}/p^n)^0 \end{tikzcd}$$
a logsmooth lift

then $R\Gamma(Z, \mathcal{O}_{Z/\mathbb{Z}/p^n}, \text{logcrys}) \simeq R\Gamma_{dR}(Z') := R\Gamma_{Z_{an}}(Z', \Omega^{\bullet, \log}_{Z'/(\mathbb{Z}/p^n)^0})$

and there is a version computing φ and N.

<u>Rq</u> : 1) The proposition is useful cause if $\mathfrak{X}$ is semistable, "étale locally étale" we look at $\mathrm{Spf}\,\mathbb{Z}_p\langle X_1 .. X_d\rangle / X_1 .. X_d - p$ whose special fiber $\mathrm{Spec}\,\mathbb{F}_p[X_1, .., X_d]/X_1 .. X_d$ has logstructure $\mathbb{N}^d$, $e_i \mapsto X_i$ and logsmooth lift $\mathrm{Spf}(\mathbb{Z}_p\langle X_1 .., X_d\rangle / X_1 .. X_d)^{X_1 .., X_d}$ over $(\mathrm{Spf}\,\mathbb{Z}_p)^0$

2) The structure $\mathbb{N} : 1 \to 0$ is important cause want some that
i) can compare to de Rham of $\mathfrak{X}$, i.e. $R\Gamma_{\text{logcrys}}(\mathfrak{X} / (\mathrm{Spf}\,\mathbb{Z}_p)^{1 \mapsto p})$ by "variation"
ii) define a Frobenius on the base, i.e. verifying $\alpha(m^p) = \varphi(\alpha(m))$. On $\mathbb{Z}_p$, the identity is a lift, by does not verify the condition for $1 \mapsto p$.

<u>Th</u> : Let X be a smooth proper $\mathbb{Q}_p$-rigid analytic variety having a semistable model $\mathfrak{X}$
1) $R\Gamma_{\text{logcrys}}(\mathfrak{X}_s)\left[\frac{1}{p}\right]$ only depend on X
2) if $\mathfrak{X}$ is smooth, it is just the crystalline cohomology.

<u>Rq</u> 1) $R\Gamma_{\text{logcrys}}(\mathbb{B}^{1,an}_{\mathbb{Q}_p})$ is infinite dimensional
2) In general, one prefer to replace logcrystalline by rigid cohomology, an overconvergent version that behaves better for affines.

3/ The model of $\mathbb{A}^{1,an}_{\mathbb{Q}_p}$

Recall we defined $\mathbb{A}^{1,an}_{\mathbb{Q}_p} = \operatorname{colim} \operatorname{Sp B} \mathbb{Q}_p\langle x_n \rangle$

As $\mathbb{B}^{1,an}_{\mathbb{Q}_p}$ has a model $\mathbb{B}^1_{\mathbb{Z}_p} := \operatorname{Spf} \mathbb{Z}_p\langle X \rangle$, the first idea is to consider $\operatorname{colim} \operatorname{Spf} \mathbb{Z}_p\langle X_n \rangle$. It is $\operatorname{Spf} \mathbb{Z}_p$ encoding that the only bounded functions on $\mathbb{A}^{1,an}_{\mathbb{Q}_p}$ are the constants.

One can also write $\mathbb{A}^{1,an}_{\mathbb{Q}_p}$ as the gluing

Hence, define $\mathcal{A}^1_{\mathbb{Z}_p}$ = gluing of $\mathbb{B}^1_{\mathbb{Z}_p}$ and $\operatorname{Spf} \mathbb{Z}_p\langle X_n, Y_n \rangle / X_nY_n - p$, semistable model!
along $X_n = Y_{n+1}^{-1}$ on $|x_n| = 1$ and $|y_{n+1}| = 1/p$

It's special fiber is

$$(\mathcal{A}^1_{\mathbb{Z}_p})_s = \text{gluing of } \mathbb{A}^1_{\mathbb{F}_p} \text{ and } \operatorname{Spec} \mathbb{F}_p[X_n, Y_n]/X_nY_n \text{ along } X_n = Y_{n+1}^{-1} \text{ on } X_n \neq 0 \text{ and } Y_{n+1} \neq 0$$

$\rightsquigarrow$  ... chain of $\mathbb{P}^1_{\mathbb{F}_p}$ with normal crossings. Call $(P_i)_{i \in \mathbb{N}}$ these irreducible components and $x_i = P_i \cap P_{i+1}$.

<u>Philosophy</u>: computations by the lift $\mathcal{A}^1_{\mathbb{Z}_p}$ is awful cause the Čech complex will have horrible terms as cohomology of affines.

4/ Weight spectral sequence

Let Z be a semistable scheme over $\mathbb{F}_p$ (locally $\mathbb{F}_p[x_1 \dots x_d]/x_1 \dots x_d$), seen as logscheme over $(\operatorname{Spec} \mathbb{F}_p)^\circ$. Call $(Z_i)_{i \in I}$ the irreducible components, who are smooth <u>schemes</u>

Def: $Z^{(d)} := \coprod_{\substack{J \subset I \\ \# J = d}} \bigcap_{j \in J} Y_j$ smooth schemes $/ \mathbb{F}_p$

Th (Nakayama)

$\exists$ spectral sequence

$$E_1^{-k, i+k} := \sum_{j \geq 0,\, j \geq -k} H^{i-2j-k}_{crys}\left(Y^{(2j+k+1)}\right) \Rightarrow H^i_{logcrys}(Z)$$

where the only maps are $(i,j,k) \mapsto (i+1, j+1, k-1)$ given by signed restriction $H^{\cdot}_{crys}(Y^{(d)}) \to H^{\cdot}_{crys}(Y^{(d+1)})$
and $(i,j,k) \mapsto (i+1, j, k-1)$ — gysin map $H^{\cdot}_{crys}(Y^{(d)}) \to H^{\cdot+2}_{crys}(Y^{(d-1)})$

for certain divisor $D \subset Z$, use $0 \to \Omega^{\cdot}_Y \to \Omega^{\cdot, \log D}_Y \xrightarrow{Res} \Omega^{\cdot}_D[-1] \to 0$

$Res(\alpha \wedge dlog f_D) = \alpha|_D$.

For $(\mathbb{A}^1_{\mathbb{Z}_p})_s$, we get

$$H^0_{crys}(\amalg x_i) \xrightarrow{gysin} H^2_{crys}(\amalg P_i)$$

$$H^1_{crys}(\amalg P_i)$$

$$H^0_{crys}(\amalg P_i) \xrightarrow{Res} H^0_{crys}(\amalg x_i)$$

lemme : $H^i_{crys}(\mathbb{P}^1_{\mathbb{F}_p}) = \begin{cases} \mathbb{Z}_p & \text{if } i = 0, 2 \\ 0 & \text{otherwise} \end{cases}$

$H^i_{crys}(\operatorname{Spec} \mathbb{F}_p) = \begin{cases} \mathbb{Z}_p & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}$

Call c, ∂_{log}, c' the basis which are canonical !

Pf hint use the smooth lift $\mathbb{P}^1_{\mathbb{Z}_p}$ and $dlog\, x \in H^1_{dR}(\mathbb{G}_m) \underset{x}{\subset} H^2_{dR}(\mathbb{P}^1)$ doesn't depend on x

lemme : $res(c) = c'$, $gysin(c) = \partial_{log}$ for any $x \in \mathbb{P}^1_{\mathbb{F}_p}$

Hence get

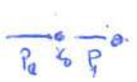
$$\prod \mathbb{Z}_p c'_i \to \prod \mathbb{Z}_p \partial_{log_i}$$

$$0$$

$$\prod \mathbb{Z}_p c_i \to \prod \mathbb{Z}_p c'_i$$

with map identifying the top row to the homology of

" the bottom row " "



Hence

Th: $H^i_{\log\text{crys}}(\mathbb{A}^1_{\mathbb{Z}_p}) = \begin{cases} \mathbb{Z}_p & \text{if } i=0 \\ 0 & \text{otherwise} \end{cases}$

Rq: 1) Can keep track of φ in the weight spectral sequence and of N. Here they're both trivial

2) This tracking of φ makes the weight spectral sequence degenerate modulo torsion when the Z_i are projective cause the lines have diagonalizable frobenii with different eigenvalues.

3) Try to do this for $\mathbb{A}^{d,an}_{\mathbb{Q}_p}$ ... the model is less easy. With A. VanHaecke we're trying to obtain a polystable weight spectral sequence to ease these computations and investigate Künneth for some Stein spaces.