LOCAL TO GLOBAL GOOD REDUCTION OF FAMILIES OF K3 SURFACES IN MIXED CHARACTERISTICS

FRANÇOIS GATINE

Contents

1. Introduction

Fix $p > 0$ a prime number, k the fraction field of a DVR $\mathcal{O}_k$ of mixed characteristic $(0, p)$. In this note we use the canonical integral model of certain orthogonal Shimura varieties developped by Madapusi Pera in [4] and [3], as well as a local-to-global result on the extension of ℓ-adic local systems to integral models by Cadoret-Tamagawa in [2], to show the following

Theorem 1.1. *Let $\mathcal{X}/\mathcal{O}_k$ be a smooth geometrically connected scheme with generic fiber X/k. If $Y \to X$ is a K3 family with primitive polarization and a certain level structure, such that each closed fiber has good reduction, then the family extends to a K3 family $\mathcal{Y} \to \mathcal{X}$.*

See Theorem 4.1 for the precise statement.

1.1. Notations.

For any scheme X we denote its set of closed points by $|X|$. For any $x \in |X|$ we denote by $k(x)$ the residue field of X at x, and $\mathcal{O}_x$ the valuation ring of $k(x)$. We let $S_x = \operatorname{Spec} \mathcal{O}_x$. A k-variety denotes a reduced separated scheme of finite type over k.

2. Shimura varieties and moduli spaces of K3 families

2.1. Recollections on K3 families.

Definition 2.1. A *K3 surface over k* is a smooth proper surface Y over k satisfying:

- $H^1(Y, \mathcal{O}_Y) = 0$, and
- the canonical bundle is trivial, i.e. $\omega_{Y/K} = \mathcal{O}_Y$.

Definition 2.2. Let B be an integral, quasi-compact scheme, let $\mathcal{Y} \to B$ be a smooth proper algebraic space over B. It is called a *K3 family* (or *relative K3 surface*) if

- each fiber is a K3 surface over the residue field, and
- the relative canonical bundle is trivial, i.e. $\omega_{\mathcal{Y}/B} = \mathcal{O}_{\mathcal{Y}}$.

If $\mathcal{Y}$ is a scheme, we call this family *schematic*.

Date: January 24, 2026.

Definition 2.3. Let B be an integral, quasi-compact scheme, let $\mathcal{Y} \to B$ be a K3 family. We say that it is *polarized (of degree d)* if there exists a relative ample sheaf $\mathcal{L}$ (of degree $d > 0$) on $\mathcal{Y}/B$. We say that it is *primitively polarized* if moreover $\mathcal{L}$ is primitive; that is, if for all geometric point $b \to B$ the line bundle $\mathcal{L}_b$ over Y_b is not a nontrivial multiple of another line bundle.

A polarized K3 family is automatically schematic.

The remainder of this section follows closely [3].

2.2. **Levels.** Denote by N the self-dual lattice $U^{\oplus 3} \oplus E_8^{\oplus 2}$ over $\mathbb{Z}$. Choose a basis (e, f) for the first copy of U in N, and for $d > 0$ set

$$L_d = \langle e - df \rangle \subseteq N$$

which is a sublattice of N of discriminant $2d$. Set $V_d := L_{d,\mathbb{Q}}$ and let $L_d^\vee \subseteq V_d$ be the dual lattice.

Remark 1. Let $Y/\mathbb{C}$ be a K3 surface over the complex numbers. Then the K3 lattice $H^2(Y(\mathbb{C}), \mathbb{Z})$ with (the negative of) its intersection pairing is isomorphic to N. A primitive polarization of degree $2d$ on Y defines a class $[\xi] \in H^2(Y, \mathbb{Z})$, and we can always find an isomorphism $H^2(Y(\mathbb{C}), \mathbb{Z}) \to N$ mapping $[\xi]$ to $e - df$, in which case the primitive cohomology is identified with L_d. Similarly, if Y/k is a K3 surface over a field k and ℓ is prime, the $\mathbb{Z}_\ell$-lattice $H^2(Y_{\overline{k}}, \mathbb{Z}_\ell)$ is isomorphic to $N_{\mathbb{Z}_\ell}$, and the isomorphism can be made to map the class of a fixed primitive polarization to $e - df$, so that the primitive cohomology corresponds to $L_{d,\mathbb{Z}_\ell}$.

Definition 2.4. The *discriminant kernel* is the largest subgroup $K_{L_d} \subset \mathrm{SO}(L_d)(\widehat{\mathbb{Z}})$ acting trivially on the discriminant $L_d^\vee / L_d$. Any compact open subgroup $K \subseteq K_{L_d}$ is called *admissible*.

From now-on we will be interested exclusively in admissible levels $K = K_p K^p \subseteq K_{L_d}$ such that $K_p = K_{L_d,p}$ (so called *hyperspecial* level structures).

2.3. **Moduli space of polarized K3 families.** Let $d > 0$ be an integer, denote $\mathcal{M}^\circ_{2d,\mathbb{Z}}$ be the moduli problem over $\mathbb{Z}$ sending a $\mathbb{Z}$-scheme T to the groupoid of tuples $(\mathcal{Y} \to T, \xi)$ composed of a K3 family with primitive polarization ξ. Let $M^\circ_{2d,\mathbb{Q}}$ be the corresponding moduli problem over $\mathbb{Q}$.

Fact 2.1 ([5], Theorem 4.3.3)**.** *$\mathcal{M}^\circ_{2d,\mathbb{Z}}$ is a separated Deligne-Mumford stack of finite type over $\mathbb{Z}$.*

For every prime ℓ the stack $\mathcal{M}^\circ_{2d,\mathbb{Z}[1/2\ell]}$ comes equipped with an ℓ-adic local system $\mathbf{H}^2_\ell$ of rank 22 corresponding to the relative second étale cohomology group of the universal polarized K3 family over $\mathcal{M}^\circ_{2d,\mathbb{Z}[1/2\ell]}$. There is a perfect symmetric Poincaré pairing on $\mathbf{H}^2_\ell$ valued in $\mathbb{Z}_\ell(-2)$, as well as a global section of the twist $\mathbf{H}^2_\ell$ defined by the Chern class $\mathrm{ch}_\ell(\xi)$. Denote by $\mathbf{P}^2_\ell$ the primitive part of the cohomology, that is, the orthogonal complement to $\mathrm{ch}_\ell(\xi)(-1)$. This is an ℓ-adic local system of rank 21 on which the the restriction pairing is perfect if $\ell \not| d$.

Over $\mathcal{M}_{2d,\mathbb{Z}_{(p)}}$ one can consider the $\widehat{\mathbb{Z}}^p$-sheaf $\mathbf{H}^2_{\widehat{\mathbb{Z}}^p} = \prod_{\ell \neq p} \mathbf{H}^2_\ell$, as well as a Chern class $\mathrm{ch}_{\widehat{\mathbb{Z}}^p}(\xi) \in \mathbf{H}^2_{\widehat{\mathbb{Z}}^p}(-1)$. Define the étale sheaf I^p over $\mathcal{M}_{2d,\mathbb{Z}_{(p)}}$ of trivializations of $\mathbf{H}^2_{\widehat{\mathbb{Z}}^p}$, i.e. such that for any scheme $B \to \mathcal{M}_{2d,\mathbb{Z}_{(p)}}$,

$$I^p(B) = \{\text{Isometries } \eta : L_d \otimes_\mathbb{Z} \underline{\widehat{\mathbb{Z}}^p} \xrightarrow{\sim} \mathbf{H}^2_{\widehat{\mathbb{Z}}^p,B}(-1) \,|\, \eta(e - df) = \mathrm{ch}_{\widehat{\mathbb{Z}}^p}(\xi)\}.$$

If $K = K_p K^p$ is an admissible level, K^p acts on the right of I^p by precomposition. A section $[\eta] \in H^0(B, I^p/K^p)$ is called a *K^p-level structure on B*. Define $\mathcal{M}^\circ_{2d,K,\mathbb{Z}_{(p)}}$ the relative moduli problem over $\mathcal{M}^\circ_{2d,\mathbb{Z}_{(p)}}$ attaching to $T \to \mathcal{M}^\circ_{2d,\mathbb{Z}_{(p)}}$ the set of level structures on B.

Fact 2.2 ([3], Proposition 2.11)**.** *$\mathcal{M}^\circ_{2d,K,\mathbb{Z}_{(p)}}$ is finite etale over $\mathcal{M}^\circ_{2d,\mathbb{Z}_{(p)}}$. For K^p small enough it is an algebraic space over $\mathbb{Z}_{(p)}$.*

We denote by $\mathbf{P}^2_{K,\ell}$ the pullback of $\mathbf{P}^2_\ell$ to $\mathcal{M}^\circ_{2d,K,\mathbb{Z}_{(p)}}$.

2.4. **Orthogonal shimura variety.** Denote $G = \mathrm{SO}(V_d)$ which is semisimple over $\mathbb{Q}$. Let X_L denote the space of oriented negative definite planes in $L_{d,\mathbb{R}}$; then the pair (G, X_L) defines a Shimura datum. We denote by Sh the Shimura variety defined by (G, X_L) with level K_{L_d}; it is a smooth Deligne-Mumford stack defined over $\mathbb{Q}$. If K is admissible, we let Sh_K be the quotient Shimura variety with level K. It defines a finite étale cover $\mathrm{Sh}_K \to \mathrm{Sh}$ over $\mathbb{Q}$ with $\mathbb{C}$-points

$$\mathrm{Sh}_K(\mathbb{C}) = G(\mathbb{Q}) \backslash (X_L \times G(\mathbb{A}_f)/K).$$

If K is neat, then Sh_K is a smooth quasi-projective variety over $\mathbb{Q}$.

Definition 2.5. We set

$$\mathrm{Sh}_{K_p} = \varprojlim_{K^p \subseteq K^p_{L_d}} \mathrm{Sh}_{K_{L_d,p}K^p}$$

where the prime-to-p component K^p is allowed to vary within the compact open subgroups of $K^p_{L_d}$.

One of the main results of [4] is that Sh_{K_p} admits a so called *smooth integral canonical model.*

Fact 2.3 ([4], Proposition 7.9)**.** *There exists a regular, formally smooth $\mathbb{Z}_{(p)}$-model $\mathscr{S}_{K_p}$ of Sh_{K_p} satisfying the following* smooth extension property*: for any regular, formally smooth $\mathbb{Z}_p$-scheme $\mathcal{R}$, any map $\mathcal{R}_{\mathbb{Q}} \to \mathrm{Sh}_{K_p}$ over $\mathbb{Q}$ extends uniquely to a map $\mathcal{R} \to \mathscr{S}_{K_p}$.*

In turn, one of the main results of [3] is a relation between the moduli space $\mathcal{M}^{\circ}_{2d,K,\mathbb{Z}_{(p)}}$ and the integral model $\mathscr{S}_K$ for some admissible levels K.

Fact 2.4 ([3], Corollary 4.15)**.** *For any neat admissible level K, there is a (non-canonical) open immersion $\rho_K : \mathcal{M}^{\circ}_{2d,K,\mathbb{Z}_{(p)}} \to \mathscr{S}_K$.*

From now-on we fix such an open immersion. It pulls back over $\mathbb{Q}$ to an open immersion

$$\rho_K : M^{\circ}_{2d,K,\mathbb{Q}} \to \mathrm{Sh}_K . \tag{1}$$

2.5. **Canonical local systems on orthogonal Shimura varieties.** For each prime $\ell \neq p$ Madapusi defines a local system $\mathbf{L}_\ell$ on $\mathscr{S}_{K_p}$ using the so-called *special endomorphisms* of a certain Kuga-Sataka construction. The pullback of $\mathbf{L}_\ell$ to the generic fiber Sh_{K_p} is also denoted $\mathbf{L}_\ell$. For every admissible level K the local system passes to the quotient and defines local systems $\mathbf{L}_{K,\ell}$ on $\mathscr{S}_K$ and Sh_K.

Fact 2.5 ([3], 3.2)**.** *For any $n \geq 1$ there exists an admissible subgroup $K(\ell^n) \subseteq K$ such that the trivializing cover of Sh_K of the finite local system $\mathbf{L}_{K,\ell}/\ell^n\mathbf{L}_{K,\ell}$ is $\mathrm{Sh}_{K(\ell^n)}$.*

Denote by $\mathbf{L}_{\ell\neq p}$ (resp. $\mathbf{L}_{K,\ell\neq p}$) the product over $\ell \neq p$ of all $\mathbf{L}_\ell$ (resp. $\mathbf{L}_{K,\ell}$) over Sh_{K_p} (resp. Sh_K).

Corollary 2.1. *For any N prime to p there exists an admissible subgroup $K(N) \subseteq K$ such that the trivializing cover of Sh_K of the finite local system $\mathbf{L}_{K,\ell\neq p}/N\mathbf{L}_{K,\ell\neq p}$ is $\mathrm{Sh}_{K(N)}$.*

The Shimura varieties Sh_K parametrize certain quadratic lattices with extra structure. From Fact 2.5 above one expects the local systems $\mathbf{L}_{K,\ell}$ to be related to such lattices; ideally $\mathbf{L}_{K,\ell}$ would be the universal $\mathbb{Z}_\ell$-lattice over Sh_K. We see from Remark 1 that the primitive cohomology of K3 surfaces precisely yields lattices of this nature. It is then a reasonable guess to assume that under the open immersion (1) the local system $\mathbf{L}_{K,\ell}$ pulls back to $\mathbf{P}^2_{K,\ell}$. This is true.

Fact 2.6 ([6], Proposition 3.9)**.** *ρ_K induces an isometry $\rho_K^*\mathbf{L}_{K,\ell} \xrightarrow{\sim} \mathbf{P}^2_{K,\ell}$ over $\mathcal{M}^{\circ}_{2d,K,\mathbb{Z}_{(p)}}$.*

Any primitively polarized K3 family over a base $B/\mathbb{Q}$ with K level structure corresponds to a map $B \to M^{\circ}_{2d,K,\mathbb{Q}}$, hence a map $B \to \mathrm{Sh}_K$. Fact 2.6 says that the pullback of $\mathbf{L}_\ell$ under this map gives back the second primitive cohomology of the polarized K3 family.

The smooth extension property in Definition 2.3, refered to as the Mine-Moonen extension property, requires "test" schemes mapping to every Sh_K for all levels $K = K_pK^p$ as in Definition 2.5. Such test schemes could for instance be pro-smooth schemes over $\mathbb{Z}_{(p)}$, which are geometrically more complicated than the finite type $\mathbb{Z}_{(p)}$ schemes. For this reason we want to use a weaker version of the smooth extension property stated in [1], in which maps between pro-objects are replaced with extensions of étale local systems.

Proposition 2.1 ([1], §1.3)**.** *Let $\mathcal{X}/\mathbb{Z}_{(p)}$ be smooth with generic fiber $X/\mathbb{Q}$. Let $\phi : X \to \mathrm{Sh}_K$ be a map, and suppose that the local system $\phi^*\mathbf{L}_{K,\ell\neq p}$ extends to $\mathcal{X}$. Then ϕ extends uniquely to a map $\mathcal{X} \to \mathscr{S}_K$.*

Proof. Let $\mathcal{V}$ be the extensions of $\phi^*\mathbf{L}_{K,\ell\neq p}$ to $\mathcal{X}$. Let N be an integer prime to p, consider $K(N)$ as in Corollary 2.1. Let $\mathcal{X}_N \to \mathcal{X}$ be the trivializing étale cover of $\mathcal{V}/N\mathcal{V}$, $X_N \to X$ its generic fiber. Then

X_N coincides with the pullback of X along the cover $\mathrm{Sh}_{K(N)} \to \mathrm{Sh}_K$, i.e. the following squares are cartesian:

$$\begin{array}{ccccc} \mathrm{Sh}_{K(N)} & \longleftarrow & X_N & \longrightarrow & \mathcal{X}_N \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Sh}_K & \longleftarrow & X & \longrightarrow & \mathcal{X} \end{array}$$

As N varies, the $K(N)$ are cofinal in all admissible levels. The projective limit of all $\mathcal{X}_N$ (resp. the generic fibers X_N) defines a regular, formally smooth scheme $\mathcal{X}_{K_p}$ (resp. its generic fiber X_{K_p}), with a map $X_{K_p} \to \mathrm{Sh}_{K_p}$. The extension property in Fact 2.3 yields a map $\mathcal{X}_{K_p} \to \mathscr{S}_{K_p}$. By looking at level K we find a map

$$\mathcal{X} \to \mathscr{S}_K.$$

Uniqueness follows from the density of X in $\mathcal{X}$. □

3. Pointwise criterion for good reduction

Let X be a smooth, geometrically connected k-variety. Let ℓ be a prime number and $\mathcal{V}$ be an ℓ-adic local system on X. If $x \in X$ is any (possibly nonclosed) point of X, denote $\mathcal{V}_x$ the pullback to the one point scheme x. We say that $\mathcal{V}_x$ is *unramified* if it extends to a local system on S_x. Define

$$|X|_\rho^{\mathrm{ur}} = \{x \in |X|\,;\mathcal{V}_x \text{ is unramified}\}.$$

To motivate Fact 3.1 below, assume that there exists a model $\mathcal{X}/\mathcal{O}_k$ of X such that $\mathcal{V}$ extends to a local system $\mathcal{V}'$ on $\mathcal{X}$. Then for any point x of X the pullback $\mathcal{V}'_{S_x}$ to S_x defines an extention of $\mathcal{V}_x$ to S_x, in particular closed points are unramified. The converse holds:

Fact 3.1 ([2] Theorem 14)**.** *For every semistable model $\mathcal{X} \to S$ of X over S one has:*

$$\mathcal{V} \text{ extends to a local system on } \mathcal{X} \iff \mathrm{Im}(\mathcal{X}(\overline{\mathcal{O}_K}) \to |X|) \subseteq |X|_\rho^{\mathrm{ur}}.$$

Example 1. Keeping X as above and $\mathcal{X} \to S$ a semistable model, we apply Fact 3.1 to the following scenario. Let $f : Y \to X$ be a smooth proper morphism, let i denote any nonnegative integer, and set $\mathcal{V} = R^i f_* \mathbb{Z}_\ell$ on X. Fact 3.1 says that if for every $x \in \mathrm{Im}(\mathcal{X}(\overline{\mathcal{O}_k}) \to |X|)$ the fiber Y_x has good reduction over S_x, then $\mathcal{V}$ extends to a local system on $\mathcal{X}$. Indeed, it follows from the smooth proper base change that $\mathcal{V}_x$ extends to a local system on S_x.

4. Statement and proof of the main Theorem

Definition 4.1. Let $Y \to X$ be a morphism of k-schemes. We say that it satisfies *pointwise good reduction* if for every $x \in |X|$ the fiber $Y_x/k(x)$ extends to a schematic relative K3 surface $\mathcal{Y}_x/\mathcal{O}_{k(x)}$.

Theorem 4.1. *Let $\mathcal{X}/\mathcal{O}_k$ be a smooth, geometrically connected scheme with generic fiber X/k. Let K be a neat admissible level. Let $Y \to X$ be a primitively polarized K3 family with K level structure, satisfying pointwise good reduction. Then the family extends to a K3 family $\mathcal{Y} \to \mathcal{X}$.*

For each $\ell \neq p$ let $\mathcal{V}_\ell$ denote the local system on X defined by the primitive cohomology of the K3 family $Y \to X$. Example 1 implies that $\mathcal{V}_\ell$ extends to a local system over $\mathcal{X}$. By Proposition 2.1 the map

$$X \to M^\circ_{2d,K,k} \to \mathrm{Sh}_{K,k}$$

extends to a map of integral models $\mathcal{X} \to \mathscr{S}_{K,\mathcal{O}_k}$. Our goal is to show that this latter map factors through the open immersion $\rho_K : \mathcal{M}^\circ_{2d,K,\mathcal{O}_k} \to \mathscr{S}_{K,\mathcal{O}_k}$. By property of open immersions it suffices to do so for closed points of $\mathcal{X}$. Let x_s be such a closed point, by smoothness and Henselian property there exists $x \in |X|$ such that x_s is the closed point in the image of the natural map $S_x \to \mathcal{X}$. The situation is summarized in the following commutative diagram:

$$\begin{array}{ccccccc} S_x & \longrightarrow & & & \mathcal{M}^\circ_{2d,K,\mathcal{O}_k} & \xrightarrow{\rho_K} & \mathscr{S}_{K,\mathcal{O}_k} \\ \uparrow & & & & \uparrow & & \uparrow \\ \mathrm{Spec}\, k(x) & \xrightarrow{x} & X & \longrightarrow & M^\circ_{2d,K,k} & \longrightarrow & \mathrm{Sh}_{K,k} \\ \downarrow & & \downarrow & & & & \downarrow \\ S_x & \longrightarrow & \mathcal{X} & \longrightarrow & & \longrightarrow & \mathscr{S}_{K,\mathcal{O}_k} \end{array}$$

The top row is defined by the pointwise good reduction assumption, while the bottoms row is the integral extension from Proposition 2.1. The middle row is the generic fiber of both the top and bottom rows. It follows from the uniqueness in Proposition 2.1 that the top and bottom maps $S_x \to \mathscr{S}_{K,\mathcal{O}_k}$ coincide. This means that the image of S_x in $\mathscr{S}_{K,\mathcal{O}_k}$ through the bottom row lies in $\mathcal{M}^\circ_{2d,K,\mathcal{O}_k}$, thus that x_s maps into $\mathcal{M}^\circ_{2d,K,\mathcal{O}_k}$ as desired.

References

[1] Benjamin Bakker, Ananth N Shankar, and Jacob Tsimerman, *Integral canonical models of exceptional shimura varieties*, arXiv preprint arXiv:2405.12392 (2024).

[2] Anna Cadoret and Akio Tamagawa, *Pointwise criteria, working draft*, To be published (2026), None.

[3] Keerthi Madapusi Pera, *The tate conjecture for k3 surfaces in odd characteristic*, Inventiones mathematicae **201** (2015), no. 2, 625–668.

[4] ———, *Integral canonical models for spin shimura varieties*, Compositio Mathematica **152** (2016), no. 4, 769–824.

[5] Jordan Rizov, *Moduli stacks of polarized k3 surfaces in mixed characteristic*, arXiv preprint math (2005).

[6] Ziquan Yang, *Isogenies between k3 surfaces over fp*, arXiv preprint arXiv:1810.08546 (2018).

Email address: `francois.gatine@imj-prg.fr`