Cohomology of homogeneous bundle via Borel-Weil-Bott

Context varieties $/k$ a field $k = \mathbb{C}$ here, but almost everything is true in finite characteristic

$G :=$ algebraic group, reductive (the maximal normal solvable subgroup has no unipotent element in the component of 1)
$\cup$
$P :=$ parabolic subgroup (a subgroup containing B)
$\cup$
$B :=$ Borel subgroup (maximal closed connected solvable subgroup)
$\cup$
$T :=$ maximal torus (maximal closed connected abelian subgroup)

Slogan: P parabolic $\iff$ G/P is smooth projective

↳ that is why we are interested in parabolic subgroup: we obtain a specific form of varieties called homogeneous

Example

(except for the maximality it is clear they satisfy the axioms of B and T)

- $G = GL_n$, $T =$ Diagonal $\begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix}$, $B =$ upper triangular $\begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}$ } with $\neq 0$ on the diagonal

For $\sum_{j=1}^{r} i_j = n$, $P = \begin{pmatrix} GL_{i_1} & & * \\ & \ddots & \\ 0 & & GL_{i_r} \end{pmatrix}$, all parabolic are of this form

- $G = Sp_{2n} = \{ M \in GL_{2n} \,/\, M^T J M = J \}$ where $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$

Here parabolic $= G \cap$ parabolic of GL_{2n} and idem for the others

$T = \left\{ \begin{pmatrix} T_1 & 0 \\ 0 & T_1^{-1} \end{pmatrix} \text{ for } T_1 \text{ diagonal of dimension } n \right\}$

$B = \left\{ \begin{pmatrix} A & C \\ 0 & (A^T)^{-1} \end{pmatrix} \,/\, A \text{ upper triangular}, A^{-1}C \text{ antisymmetric} \right\}$

$P = \left\{ \begin{pmatrix} A & C \\ 0 & (A^T)^{-1} \end{pmatrix} \,/\, A^{-1}C \text{ antisymmetric}, A \in \begin{pmatrix} GL_{i_1} & * \\ 0 & GL_{i_r} \end{pmatrix} \right\}$

- $G = SO_n$

Example

$G = GL_n \curvearrowright \mathrm{Flag}(a_1 \dots a_{r-1})$ (transitively) $= \{ V_{a_1} \subset \dots \subset V_{a_{r-1}} \text{ s.t. } \dim(V_{a_k}) = a_k \}$

if $\sum_{j=1}^{l} i_j = a_l$, the stabilizator of a specific flag in an adapted basis is

$$\begin{pmatrix} GL_{i_1} & * & * & \\ & GL_{i_2} & * & * \\ & & GL_{i_3} & \\ 0 & & & \ddots \\ & & & & GL_{i_r} \end{pmatrix} \rightsquigarrow \text{a parabolic !!}$$

So $G/P \simeq Fl(a_1 \cdots a_{r-1})$

For $G = Sp_{2n}$ or SO_n, we find flags of subspaces where the bilinear form $= 0$

Notation

$Fl(k) = \{ k \text{ vector spaces in } \mathbb{C}^n \} = Gr(k,n)$ } we recover some well known varieties
$Gr(1,n) = \mathbb{P}^{n-1}$

Définition

An homogeneous bundle on G/P is a vector bundle $\mathcal{E}$ of the form $(G \times \mathbb{C}^r)/P$ via $\rho : P \to GL_r$ a representation of P. We have a bijection
$\{ \rho : P \to GL_r \} \longleftrightarrow \{ \text{homogeneous bundle of rank } r \}$
This class is stable by $\oplus, \Lambda^k, \mathrm{Sym}^k, (\cdot)^*, \otimes \dots$

Example

On $Gr(k,n)$ we have
$0 \to U \to \mathcal{O}^{\oplus n}_{Gr(k,n)} \to Q \to 0$
(U: universal bundle of rank k)

U is homogeneous, hence Q, hence $\mathcal{O}(1) = \Lambda^{n-k} Q$, hence $\mathcal{T}_{Gr(k,n)} = U^* \otimes Q$, hence $\Omega^1 = \mathcal{T}^*_{Gr(k,n)}$, hence Ω^k etc..

Slogan: All vector bundles may not be homogeneous, but the interesting one are. In particular all line bundle are homogeneous

Because in algebraic geometry we prefer $\mathbb{P}(V^*)$ to $\mathbb{P}(V)$

Indeed $P = \begin{pmatrix} GL_k & \mathbb{A}^{k(n-k)} \\ 0 & GL_{n-k} \end{pmatrix} \simeq (GL_k \times GL_{n-k}) \ltimes \mathbb{A}^{k(n-k)}$

P has a natural projection $P \to GL_k$, it gives the bundle $U^\vee$. $P \to GL_{n-k}$ gives $Q^\vee$
(the other bundles come from natural operations)

Representation theory

For G a semisimple, or at least reductive, group, we have $\mathfrak{g} = T_e G$, $\cup$ $\mathfrak{h} = T_e T$ } Lie algebras

$e_1 \cdots e_n$ natural basis of $\mathfrak{h}$, $\varepsilon_1 \cdots \varepsilon_n$ dual basis in $\mathfrak{h}^*$

Theorem We can decompose $\mathfrak{g} \simeq \mathfrak{h} \oplus \bigoplus_{\lambda \in \mathfrak{h}^*} \mathfrak{g}_\lambda$
Where $\mathfrak{g}_\lambda = \{ g \in \mathfrak{g} \,/\, \forall h \in \mathfrak{h}, [h,g] = \lambda(h) \cdot g \}$

- The finite set of $\lambda \neq 0$ such that $\mathfrak{g}_\lambda \neq 0$ is called the root system $\phi \subset \mathfrak{h}^*$

This choice is equivalent to the choice of a borel containing T because $T_e B = \mathfrak{h} \oplus \bigoplus_{\lambda \in \phi^+} \mathfrak{g}_\lambda$

- We can decompose $\phi = \phi^+ \cup \phi^-$ ($-\phi^+$)
- $\Delta \subset \phi^+$ indecomposable elements are called irreducible roots
- if $\langle \cdot | \cdot \rangle$ such that $\langle \varepsilon_i | \varepsilon_j \rangle = \delta_{ij}$ & if $\alpha = \sum a_i \varepsilon_i$, $\alpha^\vee = \sum a_i e_i$ then $\phi^\vee = \{ \frac{2}{\langle \alpha | \alpha \rangle} \alpha^\vee \,/\, \alpha \in \phi \}$ is the coroot system
- The dual basis to $\Delta^\vee$ are the fundamental weights $\{ w_i \}$
- $C = \mathbb{N}$-span of w_i is the Weyl chamber.
- For $\Delta = \{ \alpha_1 \cdots \alpha_n \}$, $s_i(\alpha) = \alpha - 2 \frac{\langle \alpha | \alpha_i \rangle}{\|\alpha_i\|^2} \alpha_i$
$W = \langle s_i \rangle$ is the Weyl group, acting on $\mathfrak{h}^*$
(C is a fundamental domain for $W \curvearrowright$ Weight lattice)

Example

Tangent of open in affine space {
On $G = GL_n$, $T = \mathrm{Diag} \simeq \mathbb{G}_m^n$
So $\mathfrak{g} = M_n$, $\mathfrak{h} = \langle E_{ii} \rangle_{i \in [1,n]}$ elementary matrices

We have to compute eigenspaces for $[E_{ii}, -]$
$[E_{ii}, E_{jk}] = E_{ii}E_{jk} - E_{jk}E_{ii} = \begin{cases} E_{jk} & \text{if } i = j \\ -E_{jk} & \text{if } i = k \\ 0 & \text{otherwise} \end{cases}$ For $j \neq k$

So E_{jk} generates $\mathfrak{g}_\lambda$ for $\lambda = \varepsilon_j - \varepsilon_k$ $\forall j \neq k$
So $\phi = \{ \varepsilon_j - \varepsilon_k \}_{j \neq k}$ hence $\phi^+ = \{ \varepsilon_j - \varepsilon_k \}_{j<k}$ & $\Delta = \{ \varepsilon_j - \varepsilon_{j+1} \}_{j \in [1, n-1]}$
So $\Delta^\vee = \{ e_j - e_{j+1} \}_{j \leq n-1}$ hence a dual family is $\{ w_k \}_{k \leq n-1}$ for $w_k = (\underbrace{1, 1 \dots 1}_{k}, \underbrace{0 \dots 0}_{n-k})$

Technical issue: as G is not semi simple, we have to add w_n which generates $\ker((T_e GL_n)^* \to (T_e SL_n)^*)$ here $w_n = (1, \cdots, 1)$

So $C = \mathbb{N}$-span of $w_i := \{ (a_1 \cdots a_n) \,/\, a_1 \geq \cdots \geq a_n \geq 0 \}$

$s_i(\varepsilon_j) = \varepsilon_j - \langle \varepsilon_j | \varepsilon_i - \varepsilon_{i+1} \rangle (\varepsilon_i - \varepsilon_{i+1}) = \begin{cases} \varepsilon_{i+1} & \text{if } i = j \\ \varepsilon_i & \text{if } i+1 = j \\ \varepsilon_j & \text{otherwise} \end{cases}$

So $s_i \curvearrowright \{ \varepsilon_1 \cdots \varepsilon_n \}$ via $(i, i+1)$ $\forall i \leq n-1$
hence $W \simeq \mathfrak{S}_n$

Return to representation of P

The maximal torus is the same for G, P & S_P, so we can see the weight in the same space $\mathfrak{h}^*$

P is not semi simple so its representation theory is quite complicated. We restrict to irreducible representation: $\rho_{|\text{unipotent}} = 0$. P has a semi simple subgroup S_P (Levi subgroup) $\rightsquigarrow$ well behaved under weights so we have a representant in the Weyl-chamber. But we also have a non trivial representation on $\det(p)$ so we add a character which can be < 0. hence we can leave the Weyl Chamber

Theorem $\{ P \subset G \text{ parabolic} \} \updownarrow \{ \text{subset of } \Delta \}$ Δ_P; for $Gr(k,n)$ $\Delta_P = \{ \alpha_k \}$

Theorem $\{ \text{irreducible representation of } P \} \longleftrightarrow \underbrace{V}_{\text{representation of } S_P} \otimes L_{w_1}^{m_1} \cdots \otimes L_{w_k}^{m_k}$, $w_i \in \Delta_P$, $m_i \in \mathbb{Z}$

As S_P has an highest weight $\lambda \in \mathfrak{h}^*$ we can consider $\lambda + \sum m_i w_i$ as the "highest weight" of the representation of P. But this weight may not be in C

Theorem (Borel-Weil-Bott)

$G = GL(V)$ or $G = Sp(V)$ ⋮

Let $\rho = \sum_{i=1}^{n} w_i = (n, \cdots, 1)$
Let $\mathcal{E}$ be homogeneous bundle, corresponding to a representation of P with highest weight α (in the previous sense)
Let $\sigma \in W$ s.t. $\sigma(\alpha + \rho) \in C - A \cdot w_n$ in the non semi-simple case, $A \in \mathbb{N}$
let $k = l(\sigma)$ for the set s_i of generators
$H^*(\mathcal{E}, G/P) = \begin{cases} H^k(\mathcal{E}, G/P) & \text{if } \sigma(\alpha + \rho) + A \cdot w_n \in \mathring{C} \\ 0 & \text{otherwise} \end{cases}$

In the first case
$H^k(\mathcal{E}, G/P) = \underbrace{G_{\sigma(\alpha+\rho) - \rho + A \cdot w_n}}_{\text{representation of } G \text{ with highest weight}} \simeq \underbrace{\Sigma^{\sigma(\alpha+\rho) - \rho + A w_n}}_{\text{schur functor}} V^* \otimes \det^{-A}(V)$

As in the case of $H(\mathbb{P}^n, \mathcal{O}(i))$, at most one cohomology $\neq 0$

Computation of the weight for useful bundles

This generalize to any flag

Over $Gr(k,n)$ we have seen that $U^\vee$ comes from $(GL_k \times GL_{n-k}) \ltimes \mathbb{A}^{k(n-k)} \to GL_k$
The eigenvectors for the natural representation are obviously $(0, \dots, 1, 0 \dots 0)$ with weight $(0, \dots 0, 1, 0 \dots 0)$
So the weights associated to $U^\vee$ are $= (0, \dots 0, \underset{\text{position } i}{1}, 0 \dots 0)$ $i \in [1,k]$. The highest one (for lexicographic order) is λ_1

Weights of U are the opposite, with highest $-\varepsilon_k$

in the 2nd convention {
$\mathcal{O}(1) = \det(U^\vee)$ so its weight is $\sum$ weights of $U^\vee$
hence $\mathcal{O}(1)$ has weight λ_k
Similarly Q has weights $-\varepsilon_n, \cdots, -\varepsilon_{k+1}$ (highest)

- $T_{Gr(k,n)} \simeq \mathrm{Hom}(U, Q) = U^\vee \otimes Q$ so its weights are all the sum "weight from $U^\vee$" + "weight from Q"

Example

Over $\mathbb{P}^n$, $\mathcal{O}(i)$ has weight $(i, 0 \cdots 0) = \lambda$
$\rho = (n+1, n, \dots, 1)$ $\lambda + \rho = (n+i+1, n \cdots 1)$
↳ if $i \geq 0$ $\lambda + \rho \in \mathring{C} \rightsquigarrow \sigma = id$, $\sigma(\lambda+\rho) - \rho = (i, 0 \dots 0)$
So $H^*(\mathbb{P}^n, \mathcal{O}(i)) = H^0(\mathbb{P}^n, \mathcal{O}(i)) = \mathrm{Sym}^i(V^*)$
↳ if $-1-n < i < 0$, there is redundance in $\sigma(\lambda + \rho)$ so $\sigma(\lambda+\rho) \in \partial C$ So $H^*(\mathbb{P}^n, \mathcal{O}(i)) = 0$
↳ if $i \leq -1-n$ we have to sort s.t. $\sigma(\lambda + \rho) = (n \dots 1, n+1-i)$ so $l(\sigma) = n$
$\sigma(\lambda+\rho) - \rho = (-1 \cdots -1, n-i)$
hence $\sigma(\lambda+\rho) - \rho + w_n = (0, \dots 0, n+1-i) \in$ interior of a chamber
hence $H^*(\mathbb{P}^n, \mathcal{O}(i)) = \det(V)^{i-1-n} \otimes \mathrm{Sym}^{n+1+i}(V) = H^n(\mathbb{P}^n, \mathcal{O}(i))$
Because for $k > 0$ $(0, 0 \dots, -k) = -k(1 \dots 1) + k(1 \dots 1, 0)$ [$\det(V)$ representation]
but $(1 \dots 1, 0) \sim \Lambda^{n-1} V^* = V^{**} = V$
So we obtain $\mathrm{Sym}^k(V)$

coherent with the fact that $\mathfrak{g} = \mathfrak{p} + \mathfrak{g}/\mathfrak{p}$ and Tangent $\longleftrightarrow$ representation $\mathfrak{g}/\mathfrak{p}$
here $P = \begin{pmatrix} GL_k & \mathbb{A}^{k(n-k)} \\ 0 & GL_{n-k} \end{pmatrix}$ so $\mathfrak{p}$ contain all E_{ij} except for E_{ij}, $i > k$, $j < n-k$
but as we consider the dual $Gr(k, V^*)$ it is more like $\begin{pmatrix} GL_k & 0 \\ \mathbb{A}^{k(n-k)} & GL_{n-k} \end{pmatrix}$

Cohomology of $T_{Gr(k,n)} := \mathcal{T} \simeq U^\vee \otimes Q$
the weights are $\varepsilon_i - \varepsilon_j$ for $i \leq k$, $j \geq k+1$
(all are negative roots)
The highest one is $\varepsilon_1 - \varepsilon_n = \alpha$

Remark: This is "highest weight of $U^\vee$" + "highest weight of Q" but in general a $\otimes$ is not irreducible so several highest exist. Here we check that all other weights can be obtained by the action of ϕ^-

So $\alpha + \rho = (n+1 \dots 2, 0)$ is already sorted with no redundance
So $\sigma = id \rightsquigarrow H^*(Gr(k,n), \mathcal{T}) = H^0(Gr(k,n), \mathcal{T})$
$\sigma(\alpha + \rho) - \rho = (1, 0 \dots 0, -1)$
$= \underbrace{(1, 0 \dots 0)}_{V^*} + \underbrace{(0 \dots -1)}_{V \otimes \det(V)}$

So $\Sigma^{\sigma(\alpha+\rho)-\rho} V^*$ is a subrepresentation of $\mathrm{End}(V) = \mathfrak{sl}_n \oplus \mathbb{C}.\mathrm{Id}$.
But $\mathbb{C}.\mathrm{Id}$ is from weight $(0 \dots 0)$ which doesn't appear in our case hence just the $\mathfrak{sl}_n$ part remains (irreducible)

Theorem

$H^*(Gr(k,n), T_{Gr(k,n)}) = H^0 = \mathrm{End}(V)/\langle \mathrm{Id} \rangle \otimes \det(V)^{-1}$

We recover the result on $\mathbb{P}^n$ via the Euler exact sequence $0 \to \mathcal{O} \to \mathcal{O}(1)^{\oplus n+1} \to \mathcal{T} \to 0$

Example

For $\Omega^1 = \mathcal{T}^\vee = U \otimes Q^\vee$, the highest weight is $\alpha = \varepsilon_{k+1} - \varepsilon_k$
$\alpha = (0 \dots -1, 1, 0 \dots 0)$ (position k)
$\rho = (n \quad n-k+1, n-k, \dots 1)$
$\sigma = (k, k+1)$ of length 1
So $\sigma(\alpha + \rho) = \rho$ hence $\sigma(\alpha+\rho) - \rho = 0$
hence $\Sigma^{\sigma(\alpha+\rho)-\rho} V^* = \mathbb{C}$ with trivial action

Theorem

$H^*(Gr(k,n), \Omega^1) = H^1(Gr(k,n), \Omega^1) = \mathbb{C}$.

Theorem (Littlewood-Richardson rule)

if α & β are partitions $\Sigma^\alpha V \otimes \Sigma^\beta V \simeq \bigoplus_\gamma c^\gamma_{\alpha,\beta} \Sigma^\gamma(V)$
where $c^\gamma_{\alpha,\beta} = \#$ LR tableaux of skew-shape γ/α of weight β
When $\beta = (1^r)$ $c^\gamma_{\alpha,\beta} = \begin{cases} 1 & \forall \gamma \text{ obtained from } \alpha \text{ by adding } r \text{ cells in } \neq \text{ columns} \\ 0 & \text{otherwise} \end{cases}$

For $\mathrm{End}(V) \simeq \underbrace{V}_{\lambda = (1,0 \dots 0)} \otimes \underbrace{V^*}_{(1 \cdots 1, 0) = \gamma} = \Sigma^{(2,1 \dots 1, 0)} V \oplus \underbrace{\Sigma^{(1 \dots 1)} V}_{\det(V) \simeq \mathbb{C}}$
Hence $\Sigma^{(2,1 \dots 1, 0)} V \simeq \mathrm{End}(V)/\mathbb{C}.\mathrm{Id} = \mathfrak{sl}(V)$