Betti cohomology of flag manifolds

Context

Same as before, we want to understand, $X = G/P$
$H^*(X, \mathbb{Z})$, if possible as a ring
(hard problem in AG)
Here we have a cellular decomposition so $H^*(X,\mathbb{Z}) \overset{d}{\simeq} CH^*(X)$ & the additional structure (Hodge, Tate) is trivial.
↳ we recover that at most one $H^i(X, \Omega^k) \neq 0$

Example

- $G = GL_m \rightsquigarrow X = Fl(a_1 \cdots a_{r-1})$
- $G = Sp_{2m} \rightsquigarrow X = SFl(a_1 \cdots a_r) = \{V_1 \subset \cdots \subset V_r \,/\, \dim(V_i) = a_i,\ \omega_{|V_i} = 0\}$, $a_r \leq m$

here a parabolic is $P = \left\{ \begin{pmatrix} A & C \\ 0 & {}^t\!A^{-1} \end{pmatrix} \Big/ A \in \begin{pmatrix} GL_{r_1} & & * \\ 0 & \ddots & GL_{r_l} \end{pmatrix},\ A^{-1}C \text{ symmetric} \right\}$

maximal torus : $T = \left\{ \begin{pmatrix} a_1 & & & & \\ & \ddots & & & 0 \\ & & a_m & & \\ & 0 & & a_1^{-1} & \\ & & & & \ddots \ a_m^{-1} \end{pmatrix} \right\}$

- if $G = SO_m$ we have to take into account $m \equiv 0[2]$

We will relate $H^*(X, \mathbb{Z})$ with some Lie algebra data,
Earlier we have define everything for GL_m.
We now explore the case of Sp_{2m}

Example

$\mathfrak{g} = \mathfrak{sp}_{2m} = \{ M \in M_m \,/\, M^T J + J M^T = 0 \}$

$= \left\{ \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix} \Big/ B \,\&\, C \text{ symmetrics} \right\}$

So $\mathfrak{h} = \langle E_{ii} - E_{m+i, m+i} \rangle_{i \in [\![1,m]\!]}$

$\mathfrak{g}$ is generated by
$\{E_{ij} - E_{m+j\, m+i}\}_{1 \leq i \neq j \leq m} \cup \{E_{m+i,j} + E_{m+j,i}\}_{1 \leq i \neq j \leq m} \cup \{E_{m+i,i}\}_{1 \leq i \leq m}$
$\cup \{E_{i,m+j} + E_{j,m+i}\}_{1 \leq i \neq j \leq m} \cup \{E_{i,m+i}\}_{1 \leq i \leq m}$

Now we have to compute $[\overbrace{E_{ii} - E_{m+i\, m+i}}^{e_i}, -]$ on everything

- $[e_i, E_{jk} - E_{m+k,m+j}] = \begin{cases} E_{jk} - E_{m+k,m+j} & \text{if } i=j \\ -(E_{jk} - E_{m+k,m+j}) & \text{if } i=k \\ 0 & \text{otherwise} \end{cases}$

- $[e_i, E_{m+j,k} + E_{m+k,j}] = \begin{cases} -(E_{m+j,k} + E_{m+k,j}) & \text{if } i=j \text{ or } k \\ 0 & \text{otherwise} \end{cases}$

- $[e_i, E_{m+j,j}] = \begin{cases} -2E_{m+j,j} & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$

- $[e_i, E_{j,k+m} + E_{k,m+j}] = \begin{cases} E_{j,k+m} + E_{k,m+j} & \text{if } i=j \text{ or } k \\ 0 & \text{otherwise} \end{cases}$

- $[e_i, E_{j,j+m}] = \begin{cases} 2E_{j,j+m} & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$

So $\phi = \{\pm\varepsilon_j \pm \varepsilon_k\}_{j \neq k} \cup \{\pm 2\varepsilon_j\}_{j \in [\![1,m]\!]}$

$\phi^+ = \{\varepsilon_i \pm \varepsilon_j\}_{i<j} \cup \{2\varepsilon_j\} \qquad \Delta = \{\varepsilon_i - \varepsilon_{i+1}\}_{i \leq m-1} \cup \{2\varepsilon_m\}$

if $i \leq m-1$ $\quad s_i(\varepsilon_j) = \varepsilon_j - (\varepsilon_j | \varepsilon_i - \varepsilon_{i+1})(\varepsilon_i - \varepsilon_{i+1}) = \begin{cases} \varepsilon_{i+1} & \text{if } j=i \\ \varepsilon_i & \text{if } j=i+1 \\ \varepsilon_j & \text{otherwise} \end{cases}$

$s_m(\varepsilon_j) = \varepsilon_j - 2(\varepsilon_j|\varepsilon_m)\varepsilon_m = \begin{cases} -\varepsilon_m & \text{if } j=m \\ \varepsilon_j & \text{otherwise} \end{cases}$

So $s_i \circlearrowright \{\varepsilon_j\}$ via $(i, i+1)$
s_m changes the sign of ε_m
$\Big\}$ $\underline{W \simeq \mathfrak{S}_m \ltimes (\mathbb{Z}/2)^m}$

$\Delta^\vee = \{e_i - e_{i+1}\}_{i \leq m-1} \cup \{2e_n\}$

Definition

Δ_P is such that
$\mathfrak{p} = T_eP \simeq \mathfrak{h} \oplus \bigoplus_{\lambda \in \phi^-} \mathfrak{g}_\lambda \oplus \bigoplus_{\lambda \notin \Delta_P} \mathfrak{g}_\lambda$
$\cap$
$\mathfrak{g}$

Recall that to $P \subset G$ parabolic we associate $\Delta_P \subset \Delta$. (For instance if P s.t $G/P = Fl(a_1 \cdots a_{r-1})$, $\Delta_P = \{\alpha_{a_1} \cdots \alpha_{a_{r-1}}\}$)

Let now $W_P \subset W$ the subgroup generated by s_{α_i} for $\alpha_i \notin \Delta_P$

Remark

if $P = G$, $\Delta_P = \phi$, $W_P = W$
if $P = B$, $\Delta_P = \Delta$, $W = 0$
More generally, if $P \subset P'$, $W_P \subset W_{P'}$

Theorem

if $\varepsilon_1 \cdots \varepsilon_m$ is the basis of $\mathfrak{h}^*$, let I_W^+ be the ideal of $\mathbb{Z}[\varepsilon_1 \cdots \varepsilon_m]$ generated by W-invariant elements of degree > 0
Then $H^*(G/P, \mathbb{Z}) = \mathbb{Z}[\underbrace{\varepsilon_1 \cdots \varepsilon_m}_{\text{in degree } 2}]^{W_P} / I_W^+$ $\boxed{\text{as a ring}}$

(if $P \subset P'$ we have a fibration $G/P \longrightarrow G/P'$ so π^* injective
As $W_P \subset W_{P'}$, π^* is the natural inclusion in our description.)

Good for π_* but doesn't give a ring structure

Alternatively $H^*(X, \mathbb{Z})$ has a base in bijection with W/W_P such that $\overline{\omega}$ is in degree $2l(\overline{\omega})$ where $l(\overline{\omega}) :=$ shortest length of an element in $\overline{\omega}$

Corollary: - $\chi(X) = |W| / |W_P|$
- $\dim(X) =$ longest length in W/W_P
- in degree 2, $\dim(H^2(X, \mathbb{Z})) = |\Delta_P|$

Example

$X = \mathbb{P}^m \quad G = GL_{m+1}$, $W = \mathfrak{S}_{m+1}$
$W_P = \langle (i, i+1) \rangle_{i \in [\![2,m]\!]} \simeq \mathfrak{S}_m$

So $\chi(\mathbb{P}^m) = m+1$, $b_2(X) = 1$.
W/W_P is completly described by the image of 1
$W/W_P = \{ \underbrace{(i+1, i) \cdots (3,2)(2,1)}_{\text{of length } i} \}_{i \in [\![1,m]\!]}$ ($1 \longrightarrow i+1$)

So $\forall i \in [\![1,m]\!]$ $b_{2i}(X) = 1$ & $\dim(\mathbb{P}^m) = m$

if $c_1 = \sum_{i=1}^{m+1} \varepsilon_i \quad \cdots \quad c_{m+1} = \prod_{i=1}^{m} t_i$
are the elementary symmetric polynomial in $\varepsilon_1 \cdots \varepsilon_{n+1}$
we have $I_W^+ = \langle c_i \rangle$
If $d_1 \cdots d_m$ ______ in $\varepsilon_2 \cdots \varepsilon_{m+1}$
the W_P invariant subring is generated by $\varepsilon_1, d_1 \cdots d_m$
But $c_i = \sum \prod \varepsilon_k = \underbrace{\sum \prod \varepsilon_k}_{\substack{\varepsilon_1 \text{ doesn't appear} \\ = d_i}} + t_1 \underbrace{\sum \prod \varepsilon_k}_{\substack{i-1 \text{ terms without } \varepsilon_1 \\ = d_{i-1}}}$

So $c_i = d_i + t_1 d_{i-1}$
Hence $d_1 = -\varepsilon_1$ & by induction $d_i = (-1)^i \varepsilon_1^i$
& $\varepsilon_1^{m+1} = 0$ So $H^*(X, \mathbb{Z}) = \dfrac{\mathbb{Z}[\varepsilon_1]}{\varepsilon_1^{m+1}}$ $\quad \varepsilon_1 = c_1(U)$

- If $X = Gr(k, m)$ $\quad W = \mathfrak{S}_m \quad W_P = \langle (12), \cdots (k-1, k)$
$(k+1, k+2) \cdots \rangle$

hence $W_P \simeq \mathfrak{S}_k \times \mathfrak{S}_{m-k}$, $\chi(X) = \binom{m}{k}$
$b_2(X) = 1$
$W/W_P = \left\{ \sigma \in \mathfrak{S}_m \,/\, \begin{matrix} \sigma(1) < \cdots < \sigma(k) \\ \sigma(k+1) < \cdots < \sigma(m) \end{matrix} \right\}$

The longest element is
$\tau = \begin{pmatrix} 1 & \cdots & k & k+1 & \cdots & m \\ m-k+1 & & m & 1 & \cdots & m-k \end{pmatrix}$

$\tau = \underbrace{[(m-k, m-k+1) \cdots (23)(12)] \cdots [(m-1, m) \cdots (k, k+1)]}_{k \text{ boxes}}$
($m-k$ elements for $1 \longrightarrow m-k+1$; $m-k$ elements for $k \longrightarrow m$)

So $l(\tau) = k(m-k)$ hence $\dim(Gr(k,m)) = k(m-k)$

We now compute $H^4(Gr(2,4), \mathbb{Z})$
↳ $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix} = (34)(23)$ & $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} = (12)(23)$
represent class of length 2 in W/W_P.
In fact we can see they are the only one.
So $b_4(Gr(2,4)) = 2$

Theorem For k fixed $Gr(k,m) \xrightarrow{m \to +\infty} Gr(k, \infty)$
So when $m \longrightarrow \infty$ $b_r(Gr(k,m))$ becomes constant in $b_r(Gr(k,\infty))$.
In the previous example $b_4(Gr(2,\infty)) = 2$

For the ring structure of $Gr(k,m)$, recall that c_i are symmetric polynomial in $\varepsilon_1 \cdots \varepsilon_m$
a_i ______ in $\varepsilon_1 \cdots \varepsilon_k$
b_i ______ in $\varepsilon_{k+1} \cdots \varepsilon_m$

Such that $c_i = \sum_{k+l=i} a_k b_l = 0$
Remark that $\sum_{k+l=i} a_k b_l = 0 \ \forall i \iff \left(1 + \sum_{i=1}^{k} a_i\right)\left(1 + \sum_{i=1}^{m-k} b_i\right) = 1$ $\boxed{\text{in the graded sense}}$

So $H^*(Gr(k,m), \mathbb{Z}) = \dfrac{\mathbb{Z}[a_1 \cdots a_k, b_1 \cdots b_{m-k}]}{(1 + \sum a_i)(1 + \sum b_i) - 1}$ as graded ring

- $X = Fl(1, 2, \cdots m)$, $W_P = \{id\}$ so $\chi(X) = m!$
$b_2(X) = m-1$

$\tau = \begin{pmatrix} 1 & \cdots & m \\ m & \cdots & 1 \end{pmatrix}$ is the longest element

$\tau = [(m, m-1) \cdots (32)(21)][(m, m-1) \cdots (32)] \cdots [(m, m-1)(m-1, m-2)]$
$[(m, m-1)]$

$\sum_{k=1}^{m-1} k = \frac{m(m-1)}{2}$ so $\dim(X) = \frac{m(m-1)}{2}$

$H^*(X, \mathbb{Z}) = \dfrac{\mathbb{Z}[\varepsilon_1 \cdots \varepsilon_m]}{\prod_{i=1}^{m}(1 + \varepsilon_i) - 1}$ as graded ring

- $SFl(m) := LGr(m, 2m)$: $W = \mathfrak{S}_m \ltimes (\mathbb{Z}/2)^m$
$W_P \simeq \mathfrak{S}_m$
so $\chi(X) = 2^m \quad b_2(X) = 1$

First we compute I_W^+ : we must consider symmetric expressions in ε_i, but if we have one $\varepsilon_i^{\text{odd}}$, as we can change sign, it won't be stable
So $I_W^+ = \underbrace{\langle c_i \rangle}_{\text{symmetric polynomial in } \varepsilon_i^2}$

For $X = LGr(2,4)$
$H^*(X, \mathbb{Z}) = \dfrac{\mathbb{Z}[\varepsilon_1, \varepsilon_2]^{W_P}}{\varepsilon_1^2 + \varepsilon_2^2,\ \varepsilon_1^2\varepsilon_2^2} \qquad W_P : \varepsilon_1 \leftrightarrow \varepsilon_2$

$\rightsquigarrow$ degree 0 : $\boxed{1}$ is W_P-invariant
$\rightsquigarrow$ degree 2 : $\boxed{\varepsilon_1 + \varepsilon_2}$ is W_P-invariant
$\rightsquigarrow$ degree 4 : $\underbrace{\varepsilon_1^2 + \varepsilon_2^2}_{\in I_W^+}$, $\boxed{\varepsilon_1\varepsilon_2}$ are W_P-invariant

$\rightsquigarrow$ degree 6 : $\underbrace{\varepsilon_1^3 + \varepsilon_2^3, \ \varepsilon_1^2\varepsilon_2 + \varepsilon_1\varepsilon_2^2}$ are W_P-invariant
$I_W^+ \ni (\varepsilon_1 + \varepsilon_2)(\varepsilon_1^2 + \varepsilon_2^2) = (\varepsilon_1^3 + \varepsilon_2^3) + (\varepsilon_1\varepsilon_2^2 + \varepsilon_1^2\varepsilon_2)$
so they generate the same line
$\boxed{\varepsilon_1^3 + \varepsilon_2^3}$

$\rightsquigarrow$ degree 8 : $\underbrace{\varepsilon_1^4 + \varepsilon_2^4}_{\substack{(\varepsilon_1^2 + \varepsilon_2^2)^2 - 2\varepsilon_1^2\varepsilon_2^2 \\ \in I_W^+}}$, $\underbrace{\varepsilon_1^3\varepsilon_2 + \varepsilon_1\varepsilon_2^3}_{\substack{\varepsilon_1\varepsilon_2(\varepsilon_1^2 + \varepsilon_2^2) \\ \in I_W^+}}$, $\underbrace{\varepsilon_1^2\varepsilon_2^2}_{\in I_W^+}$ are W_P-invariant

No cohomological class remain.
In particular $\underline{\dim(LGr(2,4)) = 3}$