$k = \bar{k}$ $\quad \mathrm{char}(k) = p > 0$

$\rightsquigarrow$ build X/k sm. proj surface

s.t. $\mathrm{Pic}_{X/k}$ is <u>not smooth</u>

Y/ℓ sm. proper, $\ell = \bar{\ell}$

$(\cdot)$ $T \longmapsto$ line bundles on $Y \times T /$
ℓ-scheme

$\mathrm{Pic}_{Y/\ell}$ = sheafification of $(\cdot)$

$= R^1 f_* \mathbb{G}_{m,Y}$ $\qquad f: Y \to \ell$

$\mathbb{G}_{m,Y}$ étale sheaf $/Y$

representable by a gr sch

$$0 \to \mathrm{Pic}^0_{Y/\ell} \to \mathrm{Pic}_{Y/\ell} \to NS(Y) \to 0$$

proper $\qquad$ fin gen ab grp

char 0 $\qquad$ char $p > 0$

$\to \mathrm{Pic}^0_{Y/\ell}$ ab var

$\alpha_p (R) = \{ a \in R \mid a^p = 0 \}$

group scheme

$k[x]/x^p$ $\qquad$ $\alpha_p \not\subset$ ab var

$\dim(\alpha_p) = 0 \quad < \quad \dim T_e \alpha_p = 1$

$\subset \mathfrak{m}/\mathfrak{m}^2 = \langle x \rangle$

dimension	tangent space
• $(\mathrm{Pic}^0_{X/k})^\vee_{red}$ is the Albanese var of X $\underline{\underline{Alb(X)}}$ (ab var)	$T_e \mathrm{Pic}_{X/k} \cong H^1(X, \mathcal{O}_X)^*$
$$\begin{tikzcd} X \arrow[r] \arrow[dr] & Alb(X) \arrow[d, dashed, "\exists !"] \\ & A \end{tikzcd}$$ ab var	$\dim \quad = h^{0,1}$
$\dim \mathrm{Pic}_{X/k} = \dim Alb(X)$ (ab var)	
• $\dim Alb(X) = \frac{1}{2} b_1$	
b_1 = first Betti number in $\underline{\text{any}}$ Weil cohomology theory	

$\mathrm{char}(k) = 2 \qquad k = \bar{k}$

E elliptic curve $\quad 0 \neq a \in E[2]$

$\sigma : E \times E \longrightarrow E \times E \qquad (x, y) \mapsto (x + a, -y)$

fixed-point free!

$$X = E \times E / \sigma \quad \text{smooth surface.}$$

<u>Igusa</u> '55: • $X \xrightarrow{p_1} E/\langle a \rangle$ is the Albanese variety

e.c.

<u>Pf</u>: exercise

$\rightsquigarrow \quad g = \dim \mathrm{Pic}_{X/k} = 1$

• geometric argument: $h^{0,1} = 2$

$\mathbb{Z}/2\mathbb{Z} = \langle \sigma \rangle \circlearrowright E \times E \qquad X = E \times E / \langle \sigma \rangle$

$\ell \neq 2$

$$E_2^{ij} : H^i_{grp}(\mathbb{Z}/2\mathbb{Z}, H^j_{ét}(E \times E, \mathbb{Q}_\ell)) \Longrightarrow H^{i+j}(X, \mathbb{Q}_\ell)$$

Hochschild-Serre spectral sequence.

étale

$\mathcal{F}$ sheaf on $E \times E$. Then $\Gamma(X, \mathcal{F}) = \Gamma(E \times E, \mathcal{F})^\sigma$

$\angle\ E \times E \to X$ étale cover

$$\Gamma(X, \_) = \Gamma(E\times E, \_)^{\overset{\sigma}{\uparrow}\text{inv}}$$

$\llcorner$ global sections

$i>0$ $\quad H^i_{grp}(\mathbb{Z}/2\mathbb{Z}, H^j_{et}(E\times E, \mathbb{Q}_\ell)) = 0$

$\circlearrowleft$ $\cdot 2$ $\quad$ acts as 0 $\quad$ isomorphism

$$H^1(X, \mathbb{Q}_\ell) = H^1(E\times E, \mathbb{Q}_\ell)^\sigma$$
$$= (\underset{\substack{\uparrow \\ id}}{H^1(E, \mathbb{Q}_\ell)} \oplus \underset{\substack{\uparrow \\ -id}}{H^1(E, \mathbb{Q}_\ell)})^\sigma$$
$$= H^1(E, \mathbb{Q}_\ell)$$

$g = 1/2\, b_1 = 1.$

$$H^i_{grp}(\mathbb{Z}/2, H^j(E\times E, \mathcal{O})) \Rightarrow H^{i+j}(X, \mathcal{O}_X)$$

$\llcorner$ killed by 2

$\sigma \circlearrowleft \underline{\underline{H^1(E\times E, \mathcal{O})}}$ trivial

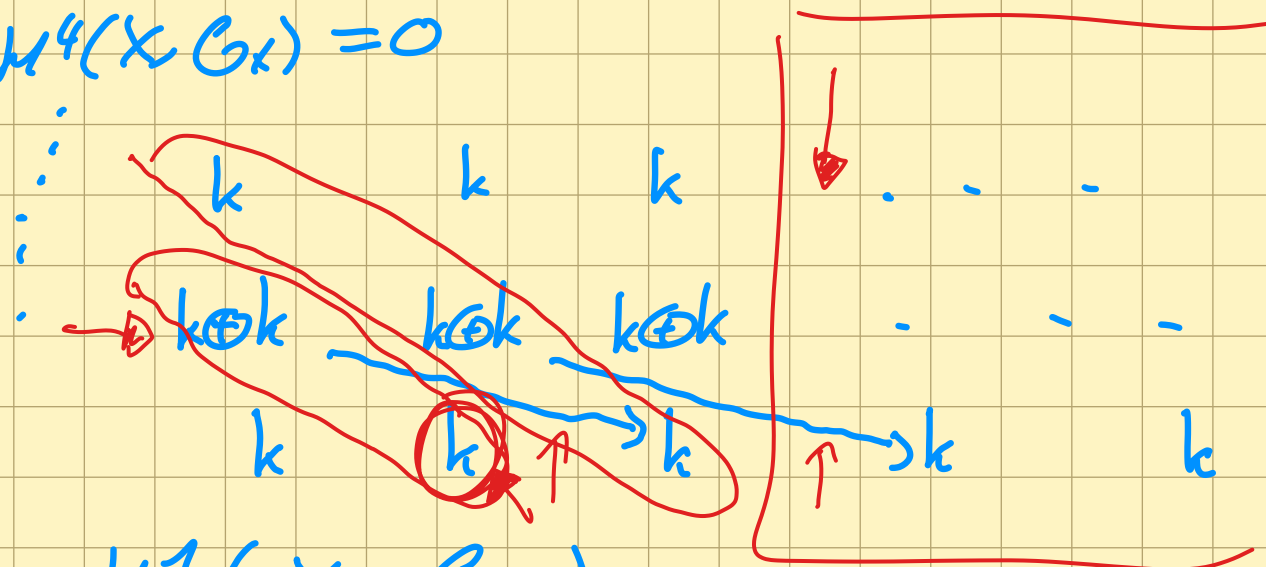
$\mathbb{Z}/2\mathbb{Z} \circlearrowleft A$ trivially, $2A = 0$

Then $\quad H^n(\mathbb{Z}/2, A) = A \quad$ (standard comp.)

$\mathbb{Z}/2\mathbb{Z}$ cohomology is 2-periodic

$H^3(X, \mathcal{O}_X) = 0$

$H^4(X, \mathcal{O}_X) = 0$



$H^1(X, \mathcal{O}_X)$

$\tau \curvearrowright Y$ $\quad$ Y/τ

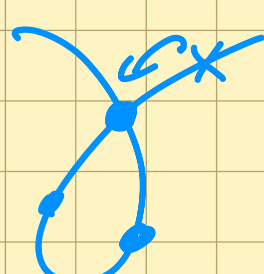
$\boxed{E \times Y/\langle\sigma\rangle}$

$\sigma : (x, y) \mapsto (x + a, \tau y)$

$$0 \to k \to H^1(X, \mathcal{O}_X) \to k \oplus k \xrightarrow{0?} k$$

$$\rightsquigarrow \dim H^1(X, \mathcal{O}_X) \geq 2$$

G grp scheme reduced $\Rightarrow$ smooth



"motivic reason"

$$X \longrightarrow \mathrm{Alb}(X)$$

$\rightsquigarrow$ iso on $h^1(-)_{\mathbb{Q}}$

$$H^1_{ét}(X, \mathbb{Z}/\ell^n(1)) = ?$$

Kümmer sequence

$$H^1_{ét}(X, \mathbb{Z}/\ell^n(1)) \simeq \mathrm{Pic}_{X/k}(k)[\ell^n]$$

$$= (\mathrm{Pic}_{X/k})_{red}(k)[\ell^n]$$

$$(\mathrm{Pic}_{X/k})_{red}(k) = \mathrm{Pic}_{X/k}(k) \;!$$