The periods of $\mathbb{P}^n$ or why you should care about motives

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These are written notes from a talk I gave for the AGES (Algebraic Geometry Examples Seminar), in which I hope to showcase the usefulness of «motivic thinking».
The question starts as follows : given an algebraic differential k-form ω on a smooth algebraic variety X, and a smooth k-cycle σ on X^{an}, we call the complex number

$$\int_\sigma \omega$$

a period. This definition is somewhat unsatisfactory, as not every element in the k-th de Rham cohomology group of an algebraic variety is the class of an algebraic k-form, but a pairing between algebraic de Rham cohomology and singular homology can be formed nonetheless.

Maybe the simplest nontrivial case of a period is with $X = \mathbb{G}_m$, $k = 1$, then the cycle should be the circle and the differential form should be $\frac{dz}{z}$. Complex analysis tells us this should evaluate to $2i\pi$. The next interesting case might be that of the projective line, or projective space in general, but we run into a problem : there are no well-defined algebraic differential forms on $\mathbb{P}^n$, so any hope of finding a genuine algebraic differential form is lost. We could (and it is a fun exercise) unwind the definition of de Rham cohomology in the case of $\mathbb{P}^n$ and compute the periods this way, but I wish to showcase another path. By embracing the more heavy and modern formalism, we can pretty efficiently compute the periods of $\mathbb{P}^n$, and then some more. In the first section, we recall the various de Rham cohomologies that exist on a smooth algebraic variety, and the comparison theorem with Betti cohomology. The second section is dedicated to computing the periods of $\mathbb{P}^n$ with some insight along the way, and the last section contains some nice bonus calculations.

1 The period isomorphism

Definition 1.1. Let M be a smooth manifold. The de Rham cohomology of M is the cohomology of the complex

$$(A^\bullet_M, d)$$

where A^k_M is the vector space of $\mathbb{C}$-valued differential forms on X and d is the exterior derivative. We denote it by $H^k_{\mathrm{dR}}(M, \mathbb{C})$ when we want to insist on the complex coefficients, or just $H^k_{\mathrm{dR}}(M)$.

Definition 1.2. Let M be a complex analytic manifold. The analytic de Rham cohomology of M is the hypercohomology

$$\mathbf{H}^k(\Omega^\bullet_M)$$

where Ω^k_M is the sheaf of holomorphic differential k-forms.
We also denote this by $H^k_{\mathrm{dR}}(M, \mathbb{C})$ or $H^k_{\mathrm{dR}}(M)$, we will soon explain why the two de Rham cohomologies we just defined are naturally isomorphic.

First, a quick refresher on hypercohomology is in order. First, recall that sheaf cohomology is constructed by taking an injective (or flasque, fine, ...) resolution $\mathcal{I}^\bullet$ of the sheaf $\mathcal{F}$ and setting

$$H^k(X, \mathcal{F}) = H^k(\mathcal{I}^\bullet(X)).$$

If we see $\mathcal{F}$ as a complex with a single term concentrated at 0, say

$$\mathcal{F}^\bullet = \cdots \to 0 \to \mathcal{F} \to 0 \to \cdots$$

then taking a resolution $\mathcal{I}^\bullet$ can be reformulated as taking a complex of injective sheaves $\mathcal{I}^\bullet$ with a map $\varepsilon : \mathcal{F}^\bullet \to \mathcal{I}^\bullet$ that induces an isomorphism in cohomology. We call such a map a quasi-isomorphism.
The procedure to define hypercohomology of a complex of sheaves $\mathcal{F}$ is then exactly this : take a complex with injective (flasque, ...) terms that is quasi-isomorphic to $\mathcal{F}^\bullet$, evaluate then compute cohomology.

We need to ensure two things : first, that injective resolutions always exist and second, that they all compute the same cohomology. The existence of a resolution can be seen by first taking a resolution of each term, then constructing a double complex with injective terms, and checking that the associated total complex is quasi-isomorphic to the original complex. The fact that all injective resolutions compute the same cohomology is very similar to the sheaf cohomology case.

In our case, we have a special resolution of $\Omega_M^\bullet$. Define, for $U \subseteq M$ open $\mathcal{A}_M^k(U)$ as the space of smooth k-forms on M, seen as a smooth manifold.

Lemma 1.3 (Poincaré lemma)**.** *The complex $\mathcal{A}_M^\bullet$ is quasi-isomorphic to its subcomplex $\mathbb{C}$ concentrated in degree 0.*

Lemma 1.4 (Holomorphic Poincaré lemma)**.** *The complex $\Omega_M^\bullet$ is quasi-isomorphic to its subcomplex $\mathbb{C}$ concentrated in degree 0.*

These lemma can be rephrased as : a closed smooth (resp. holomorphic) differential form is always locally exact, i.e. it is locally the differential of a smooth (resp. holomorphic) form. The $\mathcal{A}_M^k$, being fine sheaves, are acyclic, meaning that hypercohomology is swiftly computed as

$$\mathbf{H}^k(\mathcal{A}_M^\bullet) = H^k(A_M^\bullet).$$

Since inclusions $\mathbb{C} \hookrightarrow \Omega_M^\bullet$ and $\mathbb{C} \hookrightarrow \mathcal{A}_M^\bullet$ are quasi-isomorphisms, the inclusion $\Omega_M^\bullet \hookrightarrow \mathcal{A}_M^\bullet$ is also a quasi-isomorphism. We therefore have a canonical isomorphism

$$\mathbf{H}^k(\Omega_M^\bullet) = H^k(A_M^\bullet)$$

and the two de Rham cohomologies we defined are one and the same.

Definition 1.5. Let X be a smooth algebraic variety over a field $K \subseteq \mathbb{C}$. The de Rham cohomology of X is the hypercohomology of the de Rham complex of X :

$$H^k_{\mathrm{dR}}(X) = \mathbf{H}^k(\Omega_X^\bullet).$$

Unlike the previous two cohomologies we defined, these a K-vector spaces and not $\mathbb{C}$-vector spaces.

As expected, this is again the same de Rham cohomology.

Theorem 1.6 ([Gro66], Theorem 1'). *If X is a smooth complex algebraic variety, then*

$$H^k_{\mathrm{dR}}(X) \xrightarrow{\sim} H^k_{\mathrm{dR}}(X^{\mathrm{an}}).$$

This should be at least a little surprising, especially given that X is not at all supposed projective here !

If X a smooth algebraic variety defined over $K \subseteq \mathbb{C}$, then we can form $X_{\mathbb{C}}$ and have $H^k_{\mathrm{dR}}(X_{\mathbb{C}}) = H^k_{\mathrm{dR}}(X) \otimes_K \mathbb{C}$. Therefore, the complexification of $H^k_{\mathrm{dR}}(X)$ is $H^k_{\mathrm{dR}}(X^{\mathrm{an}})$, where we allow ourselves to write X^{an} for $X^{\mathrm{an}}_{\mathbb{C}}$.

Now that we have introduced all the de Rham players, all that is left to state the comparison isomorphism is to define Betti cohomology.

Definition 1.7. Let S be a topological space. The Betti cohomology of S, noted $H^k_{\mathrm{B}}(S,\mathbb{Q})$ or $H^k_{\mathrm{B}}(S)$ is the cohomology of the constant sheaf $\mathbb{Q}$ on S.

We will also write $H^k_{\mathrm{B}}(S,\mathbb{C})$ for Betti cohomology with complex coefficients, it is the cohomology of the constant sheaf $\mathbb{C}$ and is isomorphic to $H^k_{\mathrm{B}}(S) \otimes_{\mathbb{Q}} \mathbb{C}$.

Remark. For S a reasonable (for instance a manifold) topological space, the cohomology $H^k_{\mathrm{B}}(S,\mathbb{Q})$ is naturally the singular cohomology of S with coefficients in $\mathbb{Q}$, that is to say, it is the dual of the singular homology of X :

$$H^k_{\mathrm{B}}(X,\mathbb{Q}) = H_k(X,\mathbb{Q})^\vee.$$

For a proof of this fact, see [Spa66], corollary 8 of section 8 and corollary 7 of section 9.

Theorem 1.8. *Let M be a smooth manifold. The quasi-isomorphism $\mathbb{C} \xrightarrow{\sim} \mathcal{A}^\bullet_M$ induces an isomorphism*

$$H^k_{\mathrm{dR}}(M,\mathbb{C}) \xrightarrow{\sim} H^k_{\mathrm{B}}(X,\mathbb{C}).$$

Remark. If we recall that $H^k_{\mathrm{B}}(M)$ is the dual $H_k(M,\mathbb{Q})^\vee$ of the k-th singular homology of M, the de Rham theorem is given by integration of differential forms against cycles, i.e. to the class of a differential form ω we associate the linear form

$$\sigma \mapsto \int_\sigma \omega.$$

This only makes sense if σ is a smooth cycle and it should be checked that it does not depend on the choice of ω and σ in the homology and cohomology classes. The first item is solved by approximating continuous functions by smooth functions and the second one is a consequence of the Stokes formula.

A direct consequence of this theorem is the famous period isomorphism.

Theorem 1.9 (Period isomorphism). *Let X be a smooth algebraic variety over a field $K \subseteq \mathbb{C}$. We have a comparison isomorphism*

$$c : H^k_{\mathrm{dR}}(X) \otimes_K \mathbb{C} \xrightarrow{\sim} H^k_{\mathrm{B}}(X^{\mathrm{an}}) \otimes_{\mathbb{Q}} \mathbb{C}$$

What we just did is a very usual process : we have added extra structure on the cohomology of our space. Here, the cohomology can be thought of as a triple $(V_K, V_{\mathbb{Q}}, c)$ where V_K and $V_{\mathbb{Q}}$ are vector spaces over K and $\mathbb{Q}$ respectively and

$$c : V_K \otimes_K \mathbb{C} \xrightarrow{\sim} V_Q \otimes_{\mathbb{Q}} \mathbb{C}$$

is an isomorphism (the comparison isomorphism). This is worth making into a definition !

Definition 1.10. A $(K, \mathbb{Q})$-vector space is a triple $(V_K, V_{\mathbb{Q}}, \alpha)$ where V_K and $V_{\mathbb{Q}}$ are finite dimensional K- and $\mathbb{Q}$-vector spaces respectively and

$$\alpha : V_K \otimes_K \mathbb{C} \xrightarrow{\sim} V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$$

is an isomorphism.

Example. The simplest example of a $(K, \mathbb{Q})$-vector space is the triple $(K, \mathbb{Q}, a)$ with $a \in \mathbb{C}^\times$. When $a = 2i\pi$, we give it a special name and call it $\mathbb{Q}(-1)$.

Remark. People more familiar with Hodge structures or Galois representation might be surprised. Indeed, $\mathbb{Q}(-1)$ should respectively be the Hodge structure of dimension $\mathbb{Q}$ with the filtration $F^0 = 0, F^1 = \mathbb{Q}$ or $\mathbb{Q}_p(-1)$ should be the dual of the cyclotomic character of the $G_{\mathbb{Q}_p}$. The entire point of motives is that these objects should all be one and the same : $H^2(\mathbb{P}^1)$ is $\mathbb{Q}(-1)$, be it as a $(K, \mathbb{Q})$-vector space, a Hodge structure or a Galois representation.
Being cheeky, one can also point out that $\mathbb{Q}(-1)$ can indeed be interpreted as the cyclotomic character of the absolute Galois group $G_{\mathbb{R}}$ of the real numbers, as the complex conjugate of $2i\pi$ is $-2i\pi$.

$(K, \mathbb{Q})$-vector spaces can be made into an abelian category, with tensor products and internal homs (such a category is called a tannakian category). Everything is defined in the obvious way and there is no major difficulty. We give the definition of morphisms for the example : a morphism $f : V \to W$ of $(K, \mathbb{Q})$-vector spaces is a couple $(f_K, f_{\mathbb{Q}})$ of linear maps, $f_K : V_K \to W_K$, $f_{\mathbb{Q}} : V_{\mathbb{Q}} \to W_{\mathbb{Q}}$ making the following diagram commute :

$$\begin{array}{ccc} V_K \otimes \mathbb{C} & \xrightarrow{f_K \otimes \mathbb{C}} & W_K \otimes \mathbb{C} \\ \downarrow{\alpha_V} & & \downarrow{\alpha_W} \\ V_{\mathbb{Q}} \otimes \mathbb{C} & \xrightarrow{f_{\mathbb{Q}} \otimes \mathbb{C}} & W_{\mathbb{Q}} \otimes \mathbb{C} \end{array}$$

A few remarks are in order : first, we have exact, faithful, additive tensor functors ω_K and $\omega_{\mathbb{Q}}$ sending V to V_K and $V_{\mathbb{Q}}$ respectively. These functors are also conservative in the sense that for $f : V \to W$ a morphism, it is an isomorphism if and only if $\omega_K(f)$ or $\omega_{\mathbb{Q}}(f)$ is an isomorphism.

This turns out to be incredibly useful : when we want to prove a statement on algebraic de rham cohomology for some construction that commutes with the comparison isomorphism, it is possible to turn around and do a purely algebraic topological proof to prove the isomorphism for Betti cohomology.

Since our goal is to compute periods, we should wonder what kind of object we can expect to recover from a $(K, \mathbb{Q})$-vector space V. By choosing bases for V_K and $V_{\mathbb{Q}}$, we can get an actual period matrix $P \in \mathrm{GL}_n(\mathbb{C})$ representing the isomorphism α, but it is well defined only up to a change of basis : choosing different bases for V_K and $V_{\mathbb{Q}}$ will give the matrix BPA^{-1} with $B \in \mathrm{GL}_n(\mathbb{Q})$ and $A \in \mathrm{GL}_n(K)$ the base change matrices. Therefore, a $(K, \mathbb{Q})$-vector space gives a unique, well-defined class $P \in \mathrm{GL}_n(\mathbb{Q}) \backslash \mathrm{GL}_n(\mathbb{C}) / \mathrm{GL}_n(K)$. As one would expect, the class of this matrix uniquely determines the $(K, \mathbb{Q})$-vector space up to isomorphism because the $(K, \mathbb{Q})$-vector spaces $(K^n, \mathbb{Q}^n, P)$ and $(K^n, \mathbb{Q}^n, BPA^{-1})$ are isomorphic via the morphism (B, A).

From now on, for X a smooth algebraic variety, we write $H^k(X)$ for the $(K, \mathbb{Q})$-vector space $(H^k_{\mathrm{dR}}(X), H^k(X^{\mathrm{an}}, \mathbb{Q}), c)$.

2 Computing $H^{2k}(\mathbb{P}^n)$ as a $(K,\mathbb{Q})$-vector space

We keep our goal in mind, computing the « period matrix » of $H^{2k}(\mathbb{P}^n_K)$ (actually just a period since those have dimension 1), which we will achieve by computing $H^{2k}(\mathbb{P}^n)$ as a $(K,\mathbb{Q})$-vector space. Our first step will be the case $n=1$, easiest to compute and a good stepping stone for the general case. We will then use the result to compute $H^2(\mathbb{P}^n)$ for all n, and finally use this to compute $H^{2k}(\mathbb{P}^n)$ in all generality.

Proposition 2.1.

$$H^2(\mathbb{P}^1) = \mathbb{Q}(-1).$$

The way we go about proving this is by using the Mayer-Vietoris exact sequence with $U = \mathbb{A}^1_0$ and $V = \mathbb{A}^1_\infty$ the two affine charts on $\mathbb{P}^1$ and $U \cap V = \mathbb{G}_m$. If we accept for a moment that the Mayer-Vietoris exact sequence commutes with the comparison isomorphism, i.e. that it is an exact sequence of $(K,\mathbb{Q})$ vector spaces, then we have

$$\cdots \to H^1(\mathbb{A}^1_0) \oplus H^1(\mathbb{A}^1_\infty) \to H^1(\mathbb{G}_m) \to H^2(\mathbb{P}^1) \to H^2(\mathbb{A}^1_0) \oplus H^2(\mathbb{A}^1_\infty) \to \cdots$$

Since $H^1(\mathbb{A}^1) = H^2(\mathbb{A}^1) = 0$, we get an isomorphism of $(K,\mathbb{Q})$ vector spaces

$$H^1(\mathbb{G}_m) \xrightarrow{\sim} H^2(\mathbb{P}^1).$$

All that is left is to check that $H^1(\mathbb{G}_m) = \mathbb{Q}(-1)$, which is guaranteed by the computation

$$\int_{\mathbb{U}} \frac{dz}{z} = 2i\pi.$$

The reason why the Mayer-Vietoris commutes with the comparison isomorphism is that it is a general construction, functorial in the complex of sheaves, in hypercohomology. Explaining it takes some time, the curious reader can consult [Stacks, Tag 01E9] where it is done for a sheaf instead of a complex of sheaves but works the same way mutatis mutandis for hypercohomology. Since the sequence is functorial in the complex of sheaves and comparison isomorphism is induced by the quasi-isomorphism $\mathbb{C} \to \Omega^\bullet_X$, the Mayer-Vietoris sequence is really a sequence of $(K,\mathbb{Q})$ vector spaces.
Now that we have computed $H^2(\mathbb{P}^1)$, we can use this to compute $H^2(\mathbb{P}^n)$ for all n.

Proposition 2.2. *For $n \geqslant 2$, the embedding $j : \mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^n$ induces an isomorphism*

$$j^* : H^2(\mathbb{P}^n) \xrightarrow{\sim} H^2(\mathbb{P}^{n-1}).$$

Therefore, $H^2(\mathbb{P}^n) = \mathbb{Q}(-1)$.

This comes down to the definition of the pullback of cycles and differential forms. A morphism $f : X \to Y$ of smooth algebraic varieties induces a pushforward

$$f_* : H_k(X^{\mathrm{an}}, \mathbb{Q}) \to H_k(Y^{\mathrm{an}}, \mathbb{Q})$$

dual to the pullback

$$f^* : H^k_{\mathrm{B}}(Y, \mathbb{Q}) \to H^k_{\mathrm{B}}(X, \mathbb{Q}).$$

Considering smooth de Rham cohomology, we also have a pullback on differential forms, and it induces a pullback $f^* : H^k_{\mathrm{dR}}(Y^{\mathrm{an}}) \to H^k_{\mathrm{dR}}(X^{\mathrm{an}})$.

By unwinding the definition of the integral of a differential form ω on a smooth singular k-cycle $\sigma : \Delta^k \to X$, we can see that

$$\int_{f_*\sigma} \omega = \int_{\Delta^k} (f_*\sigma)^*\omega = \int_{\Delta^k} \sigma^* f^* \omega = \int_\sigma f^*\omega$$

hence the pullback commutes with the comparison isomorphism.

In order to understand how the pullback is defined on $H^k_{\mathrm{dR}}(X)$, we need a more hypercohomological description. Take $f : X \to Y$ a morphism of smooth algebraic varieties and $\mathcal{F}^\bullet$ a complex of sheaves on T. Choose an injective resolution $\mathcal{I}^\bullet$ of $\mathcal{F}^\bullet$: since f^{-1} is exact, an injective resolution $\mathcal{J}^\bullet$ of $f^{-1}\mathcal{I}^\bullet$ will be a resolution of $f^{-1}\mathcal{F}^\bullet$. Consider the morphism of complexes

$$\mathcal{I}^\bullet \to f_*f^{-1}\mathcal{I}^\bullet \to f_*\mathcal{J}^\bullet$$

where the first map is the unit map from the adjunction between f_* and f^{-1}. Evaluating on Y gives a map

$$\mathcal{I}^\bullet(X) \to \mathcal{J}^\bullet(X)$$

which corresponds to a morphism $\mathbf{H}^k(Y, \mathcal{F}^\bullet) \to \mathbf{H}^k(X, f^{-1}\mathcal{F}^\bullet)$.
For $\mathcal{F}^\bullet = \mathbb{Q}$ we are done since $f^{-1}\mathbb{Q} = \mathbb{Q}$. In order to get the pullback on de Rham cohomology, one should go one step further compose this with the map $f^{-1}\Omega^\bullet_Y \to \Omega^\bullet_X$ given by the pullback of differential forms. It is not immediately obvious that this does correspond to the pullback of global differential forms in de smooth description of de Rham cohomology but one can check.
We could also directly check using that description that the comparison isomorphism commutes with pullback, by explicitly writing things out and using the fact that both $f^{-1}\Omega_{Y^{\mathrm{an}}}$ and $\Omega_{X^{\mathrm{an}}}$ are resolutions of $\mathbb{C}$ on X^{an}.

All that is left to do is check that the pullback is an isomorphism in some realization, i.e. that

$$H^2(\mathbb{P}^n) \to H^2(\mathbb{P}^{n-1})$$

is an isomorphism for any cohomology (smooth de Rham, analytic de Rham, Betti,...).

We give two ways of doing this : the first one hinges on the fact that $H^2_{\mathrm{dR}}(\mathbb{P}^n, \mathbb{C})$ has dimension 1 and is generated by its Kähler form

$$\omega_n = \sum_{i=1}^{n} \frac{dz_i d\overline{z}_i}{(1 + |z_i|^2)^2}.$$

Clearly, $j^*\omega_n = \omega_{n-1}$, so j^* is an isomorphism.

Another way to prove this fact is by cellular homology : $\mathbb{CP}^n$ has a natural CW-complex structure. Indeed, consider the closed ball $\overline{B} \subseteq \mathbb{C}^n$ and the map $\overline{B} \to \mathbb{CP}^n$ given by

$$\boldsymbol{z} = (z_1, ..., z_n) \mapsto \left[\sqrt{1 - |\boldsymbol{z}|^2} : z_1 : ... : z_n\right].$$

The interior B of the ball is sent homeomorphically to $\mathbb{C}^n \subseteq \mathbb{CP}^n$, and the boundary of the ball, where $|z|^2 = 1$, is sent to $\mathbb{CP}^{n-1}$ at infinity. By iterating this construction, we find a cellular decomposition of $\mathbb{CP}^n$ where the $2k$-skeleton is $\mathbb{CP}^k$. This implies, by usual facts on cellular homology, that $j_* : H_2(\mathbb{CP}^{n-1}) \to H_2(\mathbb{CP}^n)$ is an isomorphism.

Proposition 2.3. *For all $1 \leq k \leq n$, we have a natural isomorphism*

$$H^{2k}(\mathbb{P}^n) = H^2(\mathbb{P}^n)^{\otimes k}.$$

The tool we use in order to prove this is the cup-product and wedge product on singular and de Rham cohomology. The crucial fact here is that it again commutes with the comparison theorem, i.e. that the comparison theorem interchanges cup product and wedge product, i.e. $c(\omega \wedge \eta) = c(\omega) \smile c(\beta)$.

It is, again, not immediately clear that the wedge product is defined on algebraic de Rham cohomology. To get an idea of how it should be done, notice that the wedge induces a morphism of complex of K-vector spaces

$$\Omega_X^\bullet \otimes_K \Omega_X^\bullet \to \Omega_X^\bullet.$$

This gives rise to a map $\mathbf{H}^k(\Omega_X^\bullet \otimes_K \Omega_X^\bullet) \to H^k_{\mathrm{dR}}(X)$.
Say $\mathcal{I}^\bullet$ is an injective resolution of $\Omega_X^\bullet$: even though the complex $\mathcal{I}^\bullet \otimes_K \mathcal{I}^\bullet$ does not necessarily have injective terms anymore, it is quasi-isomorphic to $\Omega_X^\bullet \otimes \Omega_X^\bullet$, because tensoring over a field is always exact. Therefore, if we choose an injective resolution $\mathcal{J}^\bullet$ of $\mathcal{I}^\bullet \otimes_K \mathcal{I}^\bullet$, it is also a resolution of $\Omega_X^\bullet \otimes_K \Omega_X^\bullet$.
Since

$$H^k\left(\mathcal{I}^\bullet(X) \otimes_K \mathcal{I}^\bullet(X)\right) = \bigoplus_{p+q=k} \mathbf{H}^p(\Omega_X^\bullet) \otimes \mathbf{H}^q(\Omega_X^\bullet)$$

by composing the maps

$$H^k\left(\mathcal{I}^\bullet(X) \otimes_K \mathcal{I}^\bullet(X)\right) \to \mathbf{H}^k(\Omega_X^\bullet \otimes \Omega_X^\bullet) \to \mathbf{H}^k(\Omega_X^\bullet)$$

we get a map

$$\bigoplus_{p+q=k} H^p_{\mathrm{dR}}(X) \otimes H^q_{\mathrm{dR}}(X) \to H^k_{\mathrm{dR}}(X)$$

induced by the wedge. It is possible, but maybe a little tedious, to check that applying this same construction to the de Rham cohomology gives the usual wedge product, so it is compatible with the isomorphism $H^k_{\mathrm{dR}}(X) \otimes \mathbb{C} = H^k_{\mathrm{dR}}(X^{\mathrm{an}})$.

This same construction, applied to $\mathbb{C}$ instead of $\Omega_X^\bullet$, gives a map

$$\bigoplus_{p+q=k} H^p_{\mathrm{B}}(X, \mathbb{C}) \otimes H^q_{\mathrm{B}}(X, \mathbb{C}) \to H^k_{\mathrm{B}}(X, \mathbb{C})$$

which corresponds to the cup product. Since multiplication $\mathbb{C} \otimes \mathbb{C} \to \mathbb{C}$ corresponds to the wedge $\Omega_X^\bullet \otimes \Omega_X^\bullet \to \Omega_X^\bullet$, we have proved that the wedge product and cup product are interchanged by the comparison isomorphism.

We now borrow a theorem from algebraic topology, namely that the ring $H^*_{\mathrm{B}}(\mathbb{P}^n, \mathbb{Q})$ is generated as a $\mathbb{Q}$-algebra by a (linear) generator of $H^2_{\mathrm{B}}(\mathbb{P}^n, \mathbb{Q})$ (see [**Hatcher**], theorem 3.19). This implies that the map

$$H^2_{\mathrm{B}}(\mathbb{P}^n, \mathbb{Q})^{\otimes k} \to H^{2k}_{\mathrm{B}}(\mathbb{P}^n, \mathbb{Q})$$

given by $\alpha_1 \otimes \cdots \otimes \alpha_n \mapsto \alpha_1 \smile \cdots \smile \alpha_n$ is an isomorphism, and same with the corresponding map

$$H^2_{\mathrm{dR}}(\mathbb{P}^n)^{\otimes k} \to H^{2k}_{\mathrm{dR}}(\mathbb{P}^n).$$

These map commute with the comparison isomorphism, yielding the announced isomorphism of $(K, \mathbb{Q})$-vector spaces.

All in all, we have proven the following :

Theorem 2.4. *For* $0 \le k \le n$,

$$H^{2k}(\mathbb{P}^n) = \mathbb{Q}(-k).$$

Before ending this section, a word of caution : since all constructions on cohomology we have seen so far seem to commute with the comparison isomorphism, one could be tempted to believe that everything always commutes with comparison isomorphisms. Here is a counterexample. For the sake of example, consider the smooth algebraic curve $\mathbb{A}^1$ and $S \subseteq \mathbb{A}^1$ finite. There exists a residue map $\mathrm{res} : H^1_{\mathrm{dR}}(\mathbb{A}^1 \setminus S) \to H^0_{\mathrm{dR}}(S)$ sending a differential form to the collection of its residues at the points of S.
The natural corresponding map on homology is the so-called tube map T, sending $[s] \in H_0(S)$ to $[\gamma_s] \in H_1(\mathbb{A}^1 \setminus S)$, the class of a small circle around s. If we denote by $T^\vee$ the corresponding map on cohomology, then the square

$$\begin{array}{ccc} H^1_{\mathrm{dR}}(\mathbb{A}^1 \setminus S, \mathbb{C}) & \xrightarrow{\mathrm{res}} & H^0_{\mathrm{dR}}(S, \mathbb{C}) \\ \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle c} \\ H^1_{\mathrm{B}}(\mathbb{A}^1 \setminus S, \mathbb{C}) & \xrightarrow{T^\vee} & H^0_{\mathrm{B}}(S, \mathbb{C}) \end{array}$$

is not commutative ! Indeed, complex analysis tells you that

$$\int_\sigma \omega = 2i\pi \sum_{s \in S} \mathrm{Ind}_s(\sigma)\mathrm{res}_s(\omega).$$

In other words, $c(\omega) = 2i\pi \sum \mathrm{res}_s(\omega)\mathrm{Ind}_s$, where Ind_s is the index around s, and the Ind_s form a basis of $H^1_{\mathrm{B}}(\mathbb{A}^1 \setminus S)$. Applying $T^\vee$, which sends Ind_s to $[s]^\vee$, gives

$$T^\vee \circ c(\omega) = 2i\pi \sum_{s \in S} \mathrm{res}_s(\omega)[s]^\vee.$$

Going the other way around the square sends ω to $\sum_{s \in S} \mathrm{res}_s(\omega)\mathbf{1}_s$ (this is the residue map), and then the comparison isomorphism between $H^0_{\mathrm{dR}}(S, \mathbb{C})$ and $H^0_{\mathrm{B}}(S, \mathbb{C})$ sends $\mathbf{1}_s$ to $[s]^\vee$. Therefore, the square is missing a $2i\pi$ factor to commute - the residue / tube map actually defines a morphism

$$H^1(\mathbb{A}^1 \setminus S) \to H^0(S) \otimes \mathbb{Q}(-1).$$

This is true in more generality : for any smooth variety X and Z smooth subvariety of codimension 1, there are a residue and a tube morphism that induce a morphism of $(K, \mathbb{Q})$-vector spaces $H^k(X) \to H^{k-1}(X)(-1)$.

Poincaré duality is another case where care is needed. Indeed, for a smooth projective variety X, Poincaré duality is obtained by the cup-product / wedge product

$$H^k(X) \otimes H^{2n-k}(X) \to H^{2n}(X).$$

The $(K, \mathbb{Q})$ vector space H^{2n} has dimension 1, but it is not necessarily trivial - see the example of $\mathbb{P}^n$, where we just computed it to be $\mathbb{Q}(-n)$. Therefore, in the case of projective space (and, spoiler, in every other case), the dual of $H^k(X)$ as a $(K, \mathbb{Q})$-vector space is $H^{2n-k}(X)(n)$.

3 Computing periods

Now that we have successfully computed the periods of $\mathbb{P}^n$, let us ride on that high and compute a few more periods.

Any smooth projective variety X of dimension n over K is a ramified covering of $\mathbb{P}^n$, i.e. there always exists a finite map

$$\pi : X \xrightarrow{m:1} \mathbb{P}^n$$

Such a map induces an isomorphism of $(K, \mathbb{Q})$-vector spaces

$$\pi^* : H^{2n}(\mathbb{P}^n) \xrightarrow{\sim} H^{2n}(X)$$

simply given by multiplication by m.

This implies, quite surprpisingly, that the integral of a top algebraic form over any smooth projective algebraic variety is always in $2\pi i K^\times$. This

The last thing we will do is compute the determinant of an matrix of elliptic integrals. Let X be an elliptic curve defined over $K \subseteq \overline{\mathbb{Q}}$ - one may for instance think of an elliptic curve in the Legendre family, given by $y^2 = x(x-1)(x-\lambda)$ with $\lambda \in K$.
Fix a basis ω, η of $H^1_{\mathrm{dR}}(X, K)$ and a basis σ, δ of $H_1(X, \mathbb{Q})$. The isomorphism $H^1_{\mathrm{dR}}(X, \mathbb{C}) \xrightarrow{\sim} H^1_{\mathrm{B}}(X, \mathbb{C})$ is given in the given bases by the matrix

$$\begin{bmatrix} \int_\sigma \omega & \int_\sigma \eta \\ \int_\delta \omega & \int_\delta \eta \end{bmatrix}.$$

Again, if X is a curve given by the equation $y^2 = x(x-1)(x-\lambda)$, everything can be made very explicit, as ω can be the class of $\frac{dx}{y}$ and η can be the class of the form with a pole with no residue $\frac{dx}{(x-\lambda)y}$. A choice of σ and δ can be explicitly made by viewing X as a covering of $\mathbb{P}^1$, ramified at $0, 1, \lambda$ and ∞.
This can be used to give a purely analytic and very concrete description of the involved integrals, which we will not present in detail here. The book [CMP17] presents all of it in great detail in its introduction. Imagine now that we want to compute the determinant of the matrix of periods. This seems like quite the complicated task a priori. However, we can notice that if we write $\alpha : H^1_{\mathrm{dR}}(X, \mathbb{C}) \to H^1_{\mathrm{B}}(X, \mathbb{C})$ the isomorphism, then the determinant we are looking for is the « matrix » of the map

$$\Lambda^2 \alpha : \Lambda^2 H^1_{\mathrm{dR}} \to \Lambda^2 H^1_{\mathrm{B}}$$

in the bases $\omega \wedge \eta$ and $\sigma \wedge \eta$.
We can compute this as follows. Since X is an elliptic curve, the space $H^1(X)$ is a $(K, \mathbb{Q})$ vector space of dimension 2, and we have a (surjective) morphism of $(K, \mathbb{Q})$ vector spaces

$$H^1(X) \otimes H^1(X) \to H^2(X)$$

given by the cup-product. Since the cup-product is anticommutative, this factors as a morphism

$$\Lambda^2 H^1(X) \to H^2(X).$$

Since both objects have dimension 1 and the morphism is not 0 by general facts on the cohomology of tori, this is an isomorphism.
Explicitly, we have an isomorphism

$$\left(\Lambda^2 H^1_{\mathrm{dR}}, \Lambda^2 H^1_{\mathrm{B}}, \Lambda^2 \alpha\right) \simeq H^2(X) = \mathbb{Q}(-1) = (K, \mathbb{Q}, 2i\pi).$$

Therefore, if the bases of H^1_{dR} and H^1_{B} are bases of $H^1_{\mathrm{dR}}(X, K)$ and $H^1_{\mathrm{B}}(X, \mathbb{Q})$ then the matrix of $\Lambda^2 \alpha$ in the bases we have chosen has to be of the form $2i\pi q$ with $q \in K^\times$

This extends to any algebraic torus T of dimension n : the morphism

$$H^1(T)^{\otimes 2n} \to H^{2n}(T)$$

is nonzero and $2n$-alternating, so it induces a morphism

$$\Lambda^{2n} H^1(T) \to H^{2n}(T) = \mathbb{Q}(-n)$$

so the determinant of a period matrix of $H^1(T)$ is always in $(2\pi i)^n K^\times$.

References

[CMP17] James Carlson, Stefan Müller-Stach, and Chris Peters. *Period Mappings and Period Domains*. Cambridge University Press, 2017.

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