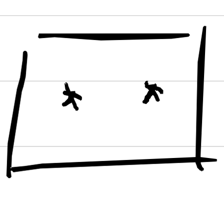
The monodromy of the Legendre family on de Rham cohomology.

$\boxed{0}$ Players

Let $B = \mathbb{P}^1_{\mathbb{C}} \setminus \{0,1,\infty\} = \mathbb{C} \setminus \{0,1\}$



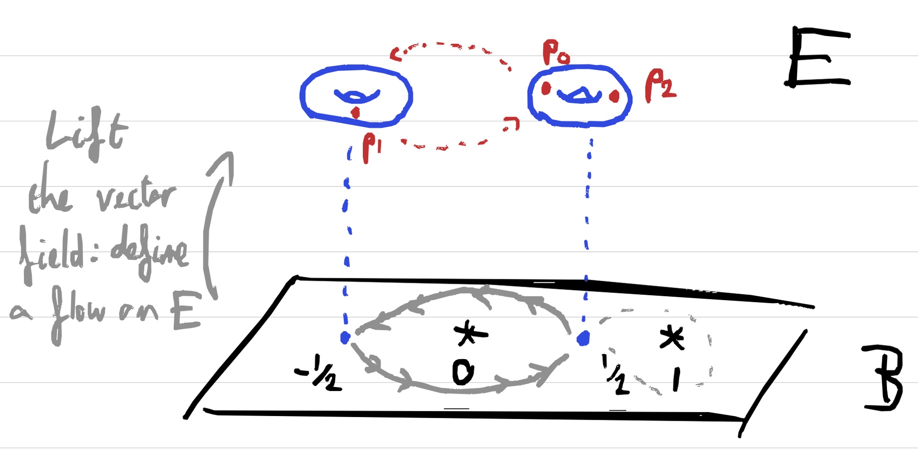
$$E = \left\{ ([x,y,z], \lambda) \in \mathbb{P}^2_{\mathbb{C}} \times B \;\middle|\; y^2 z = x(x-z)(x-\lambda z) \right\}$$

$\forall \lambda \in B$, E_λ is an elliptic curve.

We are interested in the (smooth proper) family:

$$\begin{array}{ccc} E & & ([x,y,z], \lambda) \\ \downarrow & & \downarrow \\ \mathbb{C}\setminus\{0,1\} & & \lambda \end{array}$$

"Legendre family".



Monodromy

⚠ Parallel transport depends on the lift of the vector field.

⇝ Linearize!

arbitrary

We find an action of $\pi_1(B) = F_2$ on $H^1_{dR}(E_{1/2}, \mathbb{C})$.

$\boxed{I}$ Monodromy action on $H^1_{dR}(E_\lambda, \mathbb{C})$

Fact: For $\lambda \in B$, the groups $H^1_{dR}(E_\lambda, \mathbb{C})$ form a holom. v.b on B:

$\mathcal{H}^1(E/B)$ interpreted as a loc. free sheaf of sections of the v.b. Fiber over λ: $H^1_{dR}(E_\lambda, \mathbb{C})$.

Fact: There is a natural parallel transport on $\mathcal{H}^1(E/B)$, i.e. there is a "Gauss - Manin" connection:

$$\nabla : \mathcal{H}^1(E/B) \otimes \underbrace{TB}_{\text{tangent v.b. as a sheaf}} \longrightarrow \mathcal{H}^1(E/B)$$

map of sheaves — "differentiation"

One can recover the monodromy from ∇.

Notice that: $\underbrace{TB}_{\text{as a space}} = B \times \frac{\partial}{\partial \lambda}\mathbb{C}$ — trivial line bundle

It suffices to compute

$$\nabla_{\frac{\partial}{\partial\lambda}} : \mathcal{H}^1(E/B) \longrightarrow \mathcal{H}^1(E/B)$$

map of sheaves with Leibniz rule

If $\lambda_0 \in B$, this gives a $\mathbb{C}$-lin map:

$$\nabla^{\lambda_0}_{\frac{\partial}{\partial\lambda}} \in GL(H^1(E_{\lambda_0}, \mathbb{C}))$$

Let us "define" $\nabla_{\frac{\partial}{\partial\lambda}}$. Fix $\lambda_0 \in B$. For any $[Z] \in H^1(E_{\lambda_0}, \mathbb{Z})$ any $[\alpha]$ local section of $\mathcal{H}^1(E/\mathbb{C})$ around λ_0, set

$$P_Z^{[\alpha]}(\lambda) = \int_Z [\alpha]_\lambda$$

"period function", holomorphic in λ

Then:

$\forall [Z] \in H^1(E_{\lambda_0}, \mathbb{Z})$,
$$\boxed{\left(\frac{\partial}{\partial\lambda} P_Z^{[\alpha]}\right)(\lambda_0) = \int_Z \nabla_{\frac{\partial}{\partial\lambda}}([\alpha])_{\lambda_0}}.$$

well defined bc of Betti - de Rham perfect pairing.

A local basis of $\mathcal{H}^1(E/B)$ is $([\omega], [\eta])$ where:

$$\omega = \frac{dx}{y} = \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}, \qquad \eta = \frac{x\,dx}{y}$$

It suffices to compute $\nabla_{\frac{\partial}{\partial\lambda}}([\omega])$, $\nabla_{\frac{\partial}{\partial\lambda}}([\eta])$.

$\boxed{II}$ Computation.

Fix $\lambda_0 \in B$, let $[Z] \in H^1(E_{\lambda_0}, \mathbb{Z})$: $P_Z^{[\omega]}(\lambda) = \int_Z \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}$.

$$\left(\frac{\partial}{\partial\lambda} P_Z^{[\alpha]}\right)(\lambda) = \int_Z \left(\frac{\partial}{\partial\lambda} \frac{1}{\sqrt{x(x-1)(x-\lambda)}}\right) dx = \int_Z \frac{1}{2} \frac{x(x-1)\,dx}{y^3}.$$

$\rightsquigarrow \nabla_{\frac{\partial}{\partial\lambda}}([\omega]) = \left[\frac{x(x-1)\,dx}{2y^3}\right]$. Express this in $([\omega], [\eta])$.

"Notice" that:

$$\frac{x(x-1)}{2y^3}\,dx = \frac{-1}{2(\lambda_0-1)}\omega + \frac{1}{2\lambda_0(\lambda_0-1)}\eta + d\left(-\frac{x(x-1)}{\lambda_0(\lambda_0-1)y}\right)$$

So:

$$\nabla_{\frac{\partial}{\partial\lambda}}([\omega]) = \frac{-1}{2(\lambda_0-1)}[\omega] + \frac{1}{2\lambda_0(\lambda_0-1)}[\eta] \qquad + 0$$

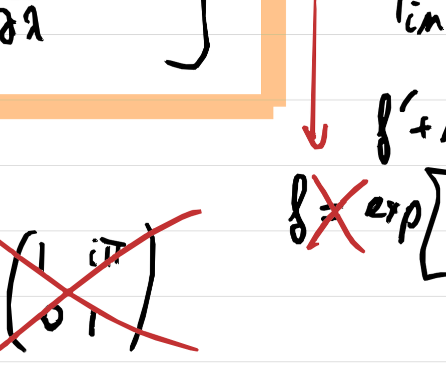
Similarly:

$$\nabla_{\frac{\partial}{\partial\lambda}}([\eta]) = \frac{-1}{2(\lambda_0-1)}[\omega] + \frac{1}{2(\lambda_0-1)}[\eta].$$

On $H^1(E_{\lambda_0}, \mathbb{C})$:

$$\boxed{\nabla^{\lambda_0}_{\frac{\partial}{\partial\lambda}} = \frac{1}{2(\lambda_0-1)} \begin{pmatrix} -1 & \frac{1}{\lambda_0} \\ -1 & 1 \end{pmatrix}}$$

We relate this to monodromy



$$\rho : \underset{\langle T_0, T_1 \rangle}{\pi_1(B)} \longrightarrow GL(H^1(E_{1/2}, \mathbb{C}))$$

See erratum below.

Then $\boxed{\rho(T_i) \cancel{=} \exp\left[-\int_{T_i} \nabla^{\zeta}_{\frac{\partial}{\partial\lambda}}\, d\zeta\right]}$

Recall 1rst order lin. diff. eq: $f' + Af = 0$, $f \cancel{=} \exp\left[-\int A\right]$

$$\rho(T_0) = \exp\left[-\begin{pmatrix} 0 & -i\pi \\ 0 & 0 \end{pmatrix}\right] = \cancel{\begin{pmatrix} 1 & i\pi \\ 0 & 1 \end{pmatrix}}$$

unipotent.

$$\rho(T_1) = \cancel{\begin{pmatrix} 1+i\pi & -i\pi \\ i\pi & 1-i\pi \end{pmatrix}}$$

acting on $H^1(E_{1/2}, \mathbb{C})$, $([\omega], [\eta])$ ($1/2$ is irrelevant)

Homework: Same with $x^3 + y^3 + z^3 - \lambda 3xyz$.

Erratum

As Thomas pointed out during the talk, the expression of the monodromy ρ in terms of the connection $\nabla_{\frac{\partial}{\partial\lambda}}$ is false in general.

The issue arises from the observation that $\nabla^{\lambda}_{\frac{\partial}{\partial\lambda}}$ does not commute with its derivative. Indeed:

$$A = \frac{1}{2(\lambda-1)} \begin{pmatrix} 1 & \frac{1}{\lambda} \\ 1 & -1 \end{pmatrix} \qquad A' = \begin{pmatrix} \frac{-1}{2(\lambda-1)^2} & \frac{-1}{2\lambda(\lambda-1)^2} + \frac{-1}{2\lambda^2(\lambda-1)} \\ \frac{-1}{2(\lambda-1)^2} & \frac{1}{2(\lambda-1)^2} \end{pmatrix}$$

$$= \frac{-1}{2(\lambda-1)^2} \begin{pmatrix} 1 & \frac{1}{\lambda} + \frac{\lambda_0 - 1}{\lambda^2} \\ 1 & -1 \end{pmatrix}$$

$$= \frac{-1}{2(\lambda-1)^2} \begin{pmatrix} 1 & \frac{2}{\lambda} - \frac{1}{\lambda^2} \\ 1 & -1 \end{pmatrix}$$

$$AA' = \frac{-1}{4(\lambda-1)^3} \begin{pmatrix} 1 + \frac{1}{\lambda} & \cdots \\ \cdots & \cdots \end{pmatrix}$$

$\neq$

$$A'A = \frac{-1}{4(\lambda-1)^3} \begin{pmatrix} 1 + \frac{2}{\lambda} - \frac{1}{\lambda^2} & \cdots \\ \cdots & \cdots \end{pmatrix}$$

Why is this important? Because for an ODE $f' + Af = 0$, over $\mathbb{R}$, say (A not constant a priori) one would like to claim that a solution is given by:

$$t \mapsto \exp\left[-\int_0^t A\right]$$

but exp only behaves well WRT differentiation if A and A' commute. Indeed:

$$(e^A)' = \sum_{n=0}^{+\infty} \frac{(A^n)'}{n!}$$

$\neq n A' A^{n-1}$ in general, unless $[A, A'] = 0$.

As this condition is not satisfied here, there is no easy formula for a solution in terms of A.
Thus, computing ρ from A is not as easy as I made it sound.

In theory, however, there are two ways of salvaging the situation. Indeed, it is true that flat sections (which give us monodromy) do satisfy (locally) the diff. equ. $f' + Af = 0$, $f : B \to \mathbb{C}^2$ locally.

- We could solve the equation along T_0 and T_1 numerically, hence find numerical approx. to $\rho(T_0)$ and $\rho(T_1)$.
- We can rearrange this equation into a lin. diff. equ. $w'' + a(\lambda) w' + b(\lambda) w = 0$, which happens to be a Fuchsian equation [for elliptic families, this equation is called the Picard-Fuchs equation].
  The theory of Fuchsian ODE is well understood, in particular their monodromy is classified. In this case, the solutions are given by so-called hypergeometric functions, and the precise matrices for monodromy are linked to special values of these functions.