Galois Representations and Motives - Exercise Sheet 2
Date: 14/03

Profinite groups

Exercise 1.1. Show that the profinite completion of groups is a functor. Namely, given $f : G \to H$ a group homomorphism, show that it induces a map between completions $\widehat{G} \to \widehat{H}$.

Exercise 1.2. Let G be a finite group, I an infinite set.

(a) Show that the profinite completion of $G^{(I)} := \bigoplus_I G$ is $G^I := \prod_I G$. Thus G^I is equipped with a profinite topology.

(b) Let A be a finite ring. Show that A^I admits finite index normal subgroups which are not open for its profinite topology. *(Hint: consider the ideal $A^{(I)}$ inside A^I)*

(c) Deduce that the canonical map $\iota : A^I \to \widehat{A^I}$ (profinite completion of A^I as an abstract group) is not continuous.

(d) Assume ι is bijective. Show that ι^{-1} is continuous. Deduce by compactness that ι is continuous, hence a contradiction.

We have shown that even though A^I is a profinite group, the canonical map to its profinite completion $\iota : A^I \to \widehat{A^I}$ is neither bijective, nor continuous.

Exercise 1.3. Show that $\widehat{\mathbb{Z}}$ is not a free abelian group. Show nevertheless that a short exact sequence of profinite groups $1 \to N \to G \to \widehat{\mathbb{Z}} \to 1$ always splits.

Exercise 1.4. Recall the theorem of Schur-Zassenhaus: extensions of finite groups of coprime order always split. Use it to show that an extension of a prime-to-p profinite group by a pro-p-group splits. *(Hint: show that a filtered limit of non-empty finite sets is nonempty)*

Exercise 1.5. Let $G = \varprojlim_n G_n$ a profinite group, μ its normalized Haar measure on its Borel σ-algebra. Let X be a subset of G of closure $\overline{X}$, let $\pi_n : G \to G_n$ the projection for all n.

(a) Show that if $\mu(\partial X) = 0$, X is measurable (for the completed Borel σ-algebra).

(b) Let $g \in G$, g_n its image in G_n for all n. Show that

$$g \in \overline{X} \iff g_n \in \pi_n(X) \ \forall n.$$

(c) Show that $\mu(\overline{X}) = \lim_{n\to+\infty} |\pi_n(X)|/|G_n|$. In particular, if $\mu(\partial X) = 0$, then $\mu(X) = \lim_{n\to+\infty} |\pi_n(X)|/|G_n|$.

Cebotarev: finite to infinite extensions

The goal of this section is to deduce the general statement of the Chebotarev density theorem from the case of finite extensions. Let K/k be a (possibly infinite) Galois extension of number fields, unramified outside a set of places $S \subseteq \Sigma_k$ of natural density 0. Let $G = \mathrm{Gal}(K/k)$.

Let $X \subseteq G$ be a union of conjugacy classes, such that $\mu(\partial X) = 0$ where μ denotes the normalized Haar measure on G. Consider a tower of finite Galois k-extensions $(K_n)_n$ such that $K = \bigcup_n K_n$. Let X_n be the image of X in $\mathrm{Gal}(K_n/k)$. Let Σ_X, Σ_{X_n} be the set of unramified places of k for which the Frobenius conjugacy class lies in X, X_n respectively.

If Σ denotes a subset of finite places of k, we let $\delta(\Sigma)$, $\delta_+(\Sigma)$, $\delta_-(\Sigma)$ denotes its natural density (if it exists), upper density and lower density respectively.

(1) Show that $(\delta(\Sigma_{X_n}))_n$ is non-increasing, and that $\lim_n \delta(\Sigma_{X_n}) \geq \delta_+(\Sigma_X)$.

(2) Show that $\lim_n \delta(\Sigma_{X_n}) = \mu(X)$. Hence we have $\mu(X) \geq \delta_+(\Sigma_X)$.

(3) Let $Y = G \setminus X$. Show that $\mu(Y) \geq \delta_+(\Sigma_Y)$.

(4) Show that $\delta_+(\Sigma_Y) = 1 - \delta_-(\Sigma_X)$.

(5) Show that $\delta(\Sigma_X)$ exists and equals $\mu(X)$.

Applications of the Cebotarev theorem

Let k be a number field. If K/k is a finite extension, we denote by $\mathrm{Spl}(K/k) \subseteq \Sigma_k$ the set of places of k that split completely in K. *(Recall that a place v of k* splits completely *over K if for all places $w|v$ of K, $e(w|v) = f(w|v) = 1$)*

Exercise 3.1. Let K/k be finite Galois. Find the density of $\mathrm{Spl}(K/k)$.

Exercise 3.2. Let K_1, K_2 be two finite Galois extensions of k. Show that the following are equivalent:

(i) $K_1 = K_2$.

(ii) $\mathrm{Spl}(K_1/k) = \mathrm{Spl}(K_2/k)$.

(iii) The sets $\mathrm{Spl}(K_1/k)$ and $\mathrm{Spl}(K_2/k)$ differ by a subset of density 0.

Exercise 3.3. Let K/k be a finite extension, and L/k the Galois closure of K. Denote $G = \mathrm{Gal}(L/k)$, $H = \mathrm{Gal}(L/K)$.

(a) Let $\mathfrak{p}$ be a prime of k, $\mathfrak{q}$ a prime of L dividing $\mathfrak{p}$. Let $D_{\mathfrak{p}}$ denote the associated decomposition subgroup of G. Show that the following is a well-defined bijection

$$\begin{aligned} H\backslash G/D_{\mathfrak{q}} &\to \Sigma_{K,\mathfrak{p}} \\ H\sigma D_{\mathfrak{q}} &\mapsto \sigma(\mathfrak{q}) \cap K \end{aligned}$$

(b) Show that $\mathrm{Spl}(K/k) = \mathrm{Spl}(L/k)$. Extend the results of 3.1 and 3.2 for non-Galois extensions.

Exercise 3.4. Let $f \in \mathcal{O}_k[T]$ be monic irreducible, and K the finite extension generated by all roots of f.

(a) Let v be a place of k which is unramified in K. Show that the factorization of f mod v is given by the cycle type of the Frobenius Frob_v at v in $\mathrm{Gal}(K/k)$ (viewed as a permutation subgroup).

(b) Show that for infinitely many places v, f mod v splits completely (*resp.* has no roots).

(c) Under which condition on $\mathrm{Gal}(K/k)$ would f mod v be irreducible for infinitely many places v ?

Leftovers

Exercise 4.1. Let k be a field, R any k-algebra (not necessarily commutative) and M a semisimple R-module of finite K-dimension. Define $R' = \mathrm{End}_R(M)$ and $R'' = \mathrm{End}_{R'}(M)$. We want to show that the natural map $R \to R''$ is surjective. Let $\phi \in R''$.

(a) Show that ϕ stabilizes every R-simple factor in the R-decomposition of M.

(b) Let S_i be a R-simple factor of M. Show that $\phi_{|S_i}$ acts as some element $a_i \in R$.

(c) Show that one can pick all a_i to be equal to some $a \in R$. Hence ϕ acts as a on M.

Exercise 4.2. Consider Galois extensions of p-adic fields $\begin{tikzcd}[column sep=small, row sep=small] & K \arrow[dl, dash] \arrow[dr, dash] & \\ U \arrow[dr, dash] & & T \arrow[dl, dash] \\ & k & \end{tikzcd}$ such that U/k is unramified, T/k is totally ramified, and $K = UT$. Show that $\mathrm{Gal}(K/U) \simeq \mathrm{Gal}(T/k)$ and $\mathrm{Gal}(K/T) \simeq \mathrm{Gal}(U/k)$.