Galois Representations and Motives - Exercise Sheet 1
Date: 10/03

Local rings, Henselian rings

Exercise 1.1. Let R be a local ring with maximal ideal $\mathfrak{m}$ and residue field κ.

(a) Let $P \in R[X]$ be monic, $A = R[X]/P$ and $\mathfrak{p}$ a maximal ideal of A. Show that $\mathfrak{p}$ contains the image of $\mathfrak{m}$.

(b) Write $\overline{P} = \prod_{i=1}^{r} \varphi_i^{a_i}$ the prime decomposition of the image of P in $\kappa[X]$. Show that the maximal ideals of A are the $\mathfrak{m}_i = (\mathfrak{m}, \Phi_i)A$, with Φ_i a lift of φ_i in $R[X]$. Show that the $\mathfrak{m}_i$s are distinct.

Exercise 1.2. We say that an integral domain R is a *discrete valuation ring (DVR)* if it satisfies one of the following descriptions.

(i) R is a local ring and a principal ideal domain, but not a field.

(ii) There is a valuation v on the field of fractions K such that $R = \{0\} \cup \{x \in K \mid v(x) \geq 0\}$.

(iii) R is an integrally closed, Noetherian local ring of Krull dimension one.

Show that these conditions are indeed equivalent.

Exercise 1.3. Let R be a Henselian valuation ring with nonarchimedean valuation v, K its field of fractions.

(a) Let $f = \sum_{i=0}^{n} a_i X^i \in K[X]$ with $a_0 a_n \neq 0$, and define $v(f) = \min_i v(a_i)$. Show that if f is irreducible then $v(f) = \min(v(a_0), v(a_n))$. In that case f monic and $a_0 \in R$ imply $f \in R[X]$.

(b) Let L/K be a finite field extension, let A be the integral closure of R in L. Show that

$$A = \{x \in L \mid \mathrm{N}_{L/K}(x) \in R\}.$$

(c) Show that v extends to a valuation w on L having A as its valuation ring. One can show, using the strong approximation theorem for nonarchimedean absolute values, that w is the unique valuation on L extending v.

We have shown that for a valuation ring R having field of fractions K, having R Henselian implies that v extends uniquely to any algebraic extension of K. The converse is true, but requires more groundwork on Newton polygons.

Structure of p-adic extensions

We fix p a prime number, k a p-adic field (finite extension of $\mathbb{Q}_p$) with valuation v, valuation ring $\mathcal{O}_k$, prime ideal $\mathfrak{m}$ and residue field k^-. The notations extend in the obvious way to any other p-adic field

Exercise 2.1. Let $n \geq 1$ and integer, and $1 \leq i \leq p-1$. Which of the following extensions are unramified, totally ramified, tamely ramified, wildly ramified, Galois ?

(a) $k(u^{1/n})/k$ with $u \in \mathcal{O}_k^\times$, n prime to p.

(b) $k(\pi_k^{1/n})/k$

(c) $(k[X]/P)/k$ with P an Eisenstein polynomial of $\mathcal{O}_k[X]$.

(d) $(k[X]/P)/k$ with $P = X^p - X - a$ irreducible, $a \in \mathcal{O}_k^\times$.

(e) $\mathbb{Q}_p(x)/\mathbb{Q}_p$ with x a root of $P = X^p - X - \frac{1}{p^i}$.

Exercise 2.2. Let K/k be a finite extension, let $e = e_{K|k}$, $f = f_{K|k}$. Consider $u_1, \dots, u_f \in \mathcal{O}_K$ algebraic integers mapping to a basis of K^-/k^-. Let π_K be a uniformizer of K. Show that $(u_i \pi_K^j)_{\substack{1 \le i \le f \\ 0 \le j \le e-1}}$ is a basis of $\mathcal{O}_K/\mathcal{O}_k$.

Exercise 2.3.

(a) Show that the compositum of unramified (*resp.* tamely ramified) extensions is again unramified (*resp.* tamely ramified).

(b) Show that the statement fails for totally ramified extensions.

(c) Show that every finite extension can be canonically decomposed into an unramified extension, followed by a (totally) tamely ramified extension, then a (totally) widly ramified extension.

Exercise 2.4. Show that k^{ur} and k^{tr} are not complete. Find non-convergent Cauchy sequences.

Ramification groups

We keep the notations above. We fix K/k a finite Galois extension, $x \in \mathcal{O}_K$ such that $\mathcal{O}_K = \mathcal{O}_k[x]$ as an $\mathcal{O}_k$-module, and define:

$$R_i := \{\sigma \in \mathrm{Gal}(K/k) \mid \sigma(a) - a \in \mathfrak{m}_K^{i+1} \ \forall a \in \mathcal{O}_K\} = \{\sigma \in \mathrm{Gal}(K/k) \mid \sigma(x) - x \in \mathfrak{m}_K^{i+1}\}.$$

We define $U_K^0 := \mathcal{O}_K^\times$ and $U_K^i := 1 + \mathfrak{m}_K^i$ if $i \ge 1$.

Exercise 3.1. Show that the two groups above defining R_i are equal. Do they agree with the definition in the lecture ?

Exercise 3.2. Define canonical injections $R_i/R_{i+1} \hookrightarrow U_i/U_{i+1}$. Define a canonical isomorphism $U_0/U_1 = (K^-)^\times$, and non-canonical isomorphisms $U_i/U_{i+1} = K^-$ for $i \ge 1$. Deduce that R_1 is a p-group, and R_0/R_1 has prime-to-p order.

Exercise 3.3.

(a) Let $K/L/k$ be an intermediate extension. Express the ramification filtration of K/L in terms of that of K/k.

(b) Let $M/K/k$ be finite Galois extensions. Can we write the ramification filtration of M/k in terms of that of K/k ?

Number fields

We fix k a number field.

Exercise 4.1. Let v be a place of k, consider an algebraic closure $\overline{k_v}$ of k_v, and let $\overline{k}$ be the algebraic closure of k inside of it. Show that there are canonical $\mathrm{Gal}(\overline{k}/k)$-equivariant bijections:

$$\{\text{Embeddings } \overline{k} \hookrightarrow \overline{k_v}\} \leftrightarrow \{\text{Places of } \overline{k}/k\} \leftrightarrow \{\text{Embeddings } \mathrm{Gal}(\overline{k_v}/k_v) \hookrightarrow \mathrm{Gal}(\overline{k}/k)\}.$$

Exercise 4.2. Let v be a place of k and K/k be a Galois extension. Show that $\mathrm{Gal}(K/k)$ acts transitively on the set of places of K dividing k.

Exercise 4.3. Consider a quadratic extension $\mathbb{Q}(\sqrt{d})/\mathbb{Q}$, d a nonzero squarefree integer. Where is it unramified ? Specify the ramification filtration of the decomposition subgroups. What are the Frobenii elements ?

Exercise 4.4. Same question with the cyclotomic extension $\mathbb{Q}(\zeta_n)/\mathbb{Q}$, $n \ge 1$.