The Space of Valuation Fans and Admissible P-Structures

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October 21, 2020

Abstract

In the whole paper K will be a *formally real field*, which means that -1 is not a finite sum of squares of elements of K, hence K has characteristic 0. As often in the literature, we shall write *real field* instead of formally real field. It is well known from Artin-Schreier theory that such fields are exactly those admitting at least one total order compatible with the field structure.

After some background in Real Algebra on orderings and valuations, we recall the notions of fans and valuation fans.

Thereafter, we introduce and study the space of valuation fans and its relations with the real spectrum of the real holomorphy ring.

Finally we provide some steps towards an abstract theory of the space of valuation fans and revisit Marshal's problem of realizability of abstract spaces of orderings.

1 Background in Real Algebra.

1.1 Preorderings, orderings.

Basic references for classical theory of real fields are [AS], [BCR], [R].

Definition 1 *A preordering T of K is a subset $T \subseteq K$, satisfying:*

$$\begin{aligned} T+T &\subseteq T,\ T\cdot T \subseteq T,\ 0,1 \in T,\ -1 \notin T \\ \text{and } T^* &= T\backslash\{0\} \text{ is a subgroup of } K^* = K\backslash\{0\}. \end{aligned}$$

Definition 2 *A preordering T is called a quadratic preordering if $K^2 \subseteq T$. If $K^{2n} \subseteq T$, T is said to be of level n. Preorderings with no level do exist.*

Zorn's lemma shows the existence of maximal quadratic preorderings; these are just the usual orderings, and are characterized by:

Definition 3 *A subset P of K is an ordering if:*

$$P + P \subseteq P,\ P \cdot P \subseteq P,\ P \cup -P = K,\ \ -1 \notin P.$$

From these properties one can deduce that $0, 1 \in P$, $P \cap -P = \{0\}$ and $\sum K^2 \subseteq P$. Here, and throughout the paper, $\sum K^{2n}$ denotes the set of all finite sums of $2n$-*th* powers.

We can also call P a positive cone: to any such ordering P one can associate a binary relation $\leq_P$. This is a total order relation compatible with the field structure, defined as follows:

$$b - a \in P \Leftrightarrow a \leq_P b.$$

Then P is the set of elements positive for the order relation $\leq_P$.

The set of orderings of a field K will be denoted by $\chi(K)$; it might also have been denoted by $SperK$ so as to coincide with the usual notation for rings.

A very nice well-known theorem from Artin-Schreier [AS] is:

Theorem 4 *Let K be a real field, $\sum K^2 = \underset{P_i \in \chi(K)}{\cap} P_i$.*

Example 5 *The field $\mathbb{R}$ admits only one ordering, and its set of positive elements is $\mathbb{R}^2$.*

Example 6 *The field $\mathbb{Q}(\sqrt[2]{2}) := \left\{a + b\sqrt[2]{2} \mid a, b \in \mathbb{Q}\right\}$ admits two orderings, one making $\sqrt[2]{2}$ positive and the other making $\sqrt[2]{2}$ negative.*

Example 7 $\mathbb{R}((X))$*, the power series field, admits also two orderings making X infinitesimal positive or negative.*

Example 8 $\mathbb{R}(X)$ *admits infinitely many orderings. For any $a \in \mathbb{R}$ one can define orderings $P_{a,+}$ and $P_{a,-}$ making $X - a$ infinitesimal positive or negative respectively. $\mathbb{R}(X)$ admits also the orderings P_+ and P_- making $\frac{1}{X}$ infinitesimal positive or negative respectively.*

1.2 Real Valuations.

The main classic references on valuations are [K], [E], [R2]; see [EP] for a more modern treatment.

Definition 9 *A Krull valuation v on a field K is a surjective map*

$$v : K^* \twoheadrightarrow \Gamma$$

where Γ is a totally ordered abelian group (called the value group), such that
(1) $v(xy) = v(x) + v(y)$ for any x, y in K^*;
(2) $v(x + y) \geq \min\{v(x), v(y)\}$, for any x, y in K^*, with $x + y$ in K^*.

The *valuation ring* of v is

$$A_v := \{x \in K \mid x = 0 \text{ or } v(x) \geq 0\}$$

and its *maximal ideal* is

$$I_v := \{x \in K \mid x = 0 \text{ or } v(x) > 0\}.$$

$k_v := A_v/I_v$ is called the *residue field* of the valuation.
$U_v := A_v \setminus I_v$ denotes the *group of units.*

Definition 10 *A valuation v on a field K is said to be real if and only if the residue field k_v is real (meaning $-1 \notin \sum k_v^2$).*

A field admits real valuations if and only if it is real. Of course a real field admits real valuations, at least the trivial one.

The converse implication follows from the Baer-Krull theorem which ensures that if k_v admits an ordering, then K admits also at least one ordering.

We now recall the definition of a valuation ring and how one can associate a valuation to a given valuation ring.

Definition 11 *A subring A of a field K is a valuation ring if for any $x \in K$, either x or x^{-1}belongs to A.*

Definition 12 *The valuation associated to a valuation ring A of K, with maximal ideal I, is given by the canonical quotient map $v : K^* \to \Gamma$, where $\Gamma := K^*/(A\backslash I)$ is ordered by $v(x) \leq v(y) \Leftrightarrow yx^{-1} \in A$.*

Example 13 *Given an ordering P in a field K, the convex hull of $\mathbb{Q}$ in K is:*

$$A(P) := \{x \in K \mid \exists r \in \mathbb{Q} \ \ r \pm x \in P\}.$$

$A(P)$ is a valuation ring in K with unique maximal ideal:

$$I(P) := \left\{x \in K \mid \forall r \in \mathbb{Q}^{+*} \ \ r \pm x \in P\right\}.$$

where $\mathbb{Q}^{+*} = \{r \in \mathbb{Q} \mid r > 0\}$.

$A(P)$ is clearly a subring of K; it is a valuation ring because $b \notin A(P)$ implies $b^{-1} \in A(P)$: let $b \notin A(P)$, assume $b > 0$, since $b \notin A(P)$ we have in particular $1 < b$, therefore $0 < b^{-1} < 1$ which implies that $b^{-1} \in A(P)$ because $A(P)$ is convex in K with respect to P.

1.3 Compatibility of an ordering with a valuation.

For this part we can refer to [Be2] and [L]. There is also a more recent book [EP].

Definition 14 *A quadratic preordering T in a field K is said to be fully compatible with a valuation v if and only if $1 + I_v \subset T$.*

In this case T induces on the residue field k_v a quadratic preordering $\overline{T}$. This pushdown preordering $\overline{T}$ is defined to be the image of $T \cap A_v$ under the natural map from the valuation ring A_v to the residue field k_v.

In the case of an ordering P, we just say that P is *compatible* with v; then $\overline{P}$, induced by P on the residue field k_v, is an ordering of k_v. Clearly $\overline{P}$ is closed under addition and multiplication and $\overline{P} \cup -\overline{P} = k_v$. If -1 was in $\overline{P}$ we would have $-1 = \overline{a}$ for some $a \in P \cap A(P)$. Then $1 + a \in I(P)$, hence $-a \in 1 + I(P) \subset P$, so we would get $a = 0$ which is impossible.

Example 15 *The trivial valuation, sending every non-zero element of K to 0, is compatible with any ordering of K.*

Proposition 16 *The valuation v associated to an ordering P of K with valuation ring*

$$A(P) := \{x \in K \mid \exists r \in \mathbb{Q} \quad r \pm x \in P\}$$

is compatible with P and pushes down on the residue field an (archimedean) ordering, hence this valuation v is real.

Proof. $I(P) := \{x \in K \mid \forall r \in \mathbb{Q}^{+*} \quad r \pm x \in P\}$ being the maximal ideal of $A(P)$ we have $1 + I(P) \subset P$. Hence the valuation is compatible with P. Then $\overline{P}$ induced by P on the residue field k_v is an archimedean ordering; we already know that $\overline{P}$ is an ordering, this ordering $\overline{P}$ is archimedean: for any $x \in A(P)$ there exists some $r \in \mathbb{Z}$ such that $-r <_P x <_P r$, hence in the residue field we have $-r <_{\overline{P}} \overline{x} <_{\overline{P}} r$, and therefore $\overline{P}$ is an archimedean ordering of k_v.

Theorem 17 *Let P be an ordering of K, and v be a valuation on K; the following are equivalent:*

(1) v is compatible with P (i.e. $1 + I_v \subset P$).

(2) $0 <_P a \leq_P b \Rightarrow v(a) \geq v(b)$ in Γ (the value group of v).

(3) The valuation ring A_v is convex in K with respect to P.

(4) The maximal ideal I_v of A_v is convex in K with respect to P

Proof. (2) $\Rightarrow$ (3) A_v convex in K means that if $x <_P y <_P z$, with $x, z \in A_v$ then $y \in A_v$, or equivalently $0 <_P a <_P b$ with $b \in A_v$ implies $a \in A_v$.

From (2) we deduce that $v(a) \geq v(b) \geq 0$ in Γ hence $a \in A_v$.

(3) $\Rightarrow$ (4) Assume $0 <_P a <_P b$ with $b \in I_v$ then $0 <_P b^{-1} <_P a^{-1}$. Since $b^{-1} \notin A_v$ using (3) we deduce $a^{-1} \notin A_v$, hence $a \in I_v$, I_v being the ideal of non invertible elements of A_v.

(4) $\Rightarrow$ (1) Let $m \in I_v$, if $1+m \notin P$ then $1+m \in -P$, so $1+m <_P 0$ hence $0 <_P 1 <_P -m$. Using the convexity of I_v in K for P, since $-m \in I_v$ too, this yields $1 \in I_v$ which is impossible.

(1) $\Rightarrow$ (2) Assume $0 <_P a \leq_P b$ and $v(a) < v(b)$ in Γ; then we deduce $0 < v(b) - v(a) = v(\frac{b}{a})$, hence $\frac{b}{a} \in I_v$, and also $-\frac{b}{a} \in I_v$ and $a \neq b$. From (1) we get $1 + (-\frac{b}{a}) \in P$, so $\frac{a-b}{a} >_P 0$, hence $a >_P b$ which is impossible.

Theorem 18 *Let $\mathcal{F}$ be the family of all valuation rings of K compatible with a given ordering P, then:*

(1) the valuation rings in $\mathcal{F}$ form a chain under inclusion;

(2) the smallest element of $\mathcal{F}$ is $A(P)$.

Proof. (1) Suppose $A, B \in \mathcal{F}$ and $A \nsubseteq B$, let $a \in A \backslash B$ and $a > 0$. We prove that $B \subset A$. Consider $0 < b \in B$, by the convexity of B in K we cannot have $0 < a \leq b$, so we must have $0 < b \leq a$. From the convexity of A in K, we deduce $b \in A$.

(2) Let $A \in \mathcal{F}$, A is convex in K and contains $\mathbb{Z}$, hence A contains $A(P)$ the convex hull of $\mathbb{Q}$ in K.

Note that any subring of K containing a valuation ring must itself be a valuation ring, hence $\mathcal{F}$ consists of all subrings of K containing $A(P)$. Remark also that $A \subset A'$ implies $I' \subset I$.

Definition 19 *The place λ_v, associated to a valuation v of K, is the map $\lambda_v : K \to k_v \cup \{\infty\}$ defined by $\lambda_v(a) = \overline{a} = a + I_v$ if $a \in A_v$, and $\lambda_v(a) = \infty$ if $a \notin A_v$.*

2 Fans and valuation fans.

In this section we mainly follow the notations and proofs of [L].

2.1 Quadratic preorderings.

The compatibility of a quadratic preordering with a valuation can be of two types. Given T a quadratic preordering in a real field K, v a valuation on K is *compatible* with T if it is compatible with *some* ordering P containing T. v is called *fully compatible* with T if it is compatible with *every* ordering P containing T. Below we give alternative characterizations.

Definition 20 *Given T a quadratic preordering in a real field K, and v a valuation on K with unique maximal ideal I_v in the associated valuation ring A_v:*

(1) v is fully compatible with T if and only if $1 + I_v \subset T$.

(2) v is compatible with T if and only if $(1 + I_v) \cap -T = \emptyset$.

(3) v is compatible with T if and only if $\overline{T}$ is a preordering in the residue field k_v.

We denote $\chi_{/T} := \{P \textit{ ordering} \mid P \supset T\}$.

A way of building fully compatible preorderings is to use the "wedge product" introduced in 1978 by Becker in [Be1] and Brocker in [BeBr].

Definition 21 *Let K be a real field, let A be a valuation ring in K, and $\pi : A \longrightarrow k_v$ be the projection map. Let T be a preordering of K and let S be a preordering of k_v such that $S \supset \overline{T}$. The wedge product is defined by $T \wedge S := T \cdot \pi^{-1}(S \backslash \{0\})$.*

We refer the reader to Lam's book ([L], p.21) to verify that $T \wedge S$ is a preordering in K, fully compatible with v, and such that residually $\overline{T \wedge S} = S$.

Again refering to [L] (3.3 p.22), remark that the wedge product $T \wedge S$ can also be defined for S a preordering of k_v and $T = T^* \cup \{0\}$ where $T^*$is a subgroup of K^*. Then $T \wedge S$ is a preordering in K, and if $K^2 \subseteq T$ then $T \wedge S$ is a quadratic preordering.

There is also an alternative definition for the wedge product:

$T \wedge S = \cap \left\{\text{orderings } P \ \mid \ P \supset T \text{ and } \overline{P} \in \chi_{/S}\right\}$

2.2 Fans of level 1.

In the context of preorderings fans were first presented by Becker and Köpping in [BK].

Definition 22 *Let K be a real field and let T be a quadratic preordering in K. T is a fan if and only if for any $S \supset T$, such that $-1 \notin S$ and such that $S^* = S \backslash \{0\}$ is a subgroup of K^* satisfying $[K^* : S^*] = 2$, S is an ordering in K.*

Note that if T is a fan any preordering containing T is again a fan.

There is an alternative useful characterization of a fan given in [L] (p.40), with proof of equivalence:

Proposition 23 *A preordering T is a fan if and only if for any $a \in K^* \backslash -T$ we have $T + aT \subset T \cup aT$. Such an element a is said to be T-rigid.*

First examples of fans are the trivial fans : these are orderings P and intersection of two orderings $P_1 \cap P_2$.

Another example is the pullback $\widehat{S}$ of a trivial fan S in k_v. Namely $\widehat{S} = K^2 \wedge S = K^2 \cdot \pi^{-1}(S \backslash \{0\})$ is a fan in K. In fact Bröcker's trivialization theorem given later says that all fans arise in this way.

Fans are well behaved for compatibility with real valuations.

Theorem 24 *Let K be a real field, v a valuation on K, and T a preordering in K. Then the followings hold:*

(a) If v is compatible with T, T is a fan implies that $\overline{T}$ is a fan in k_v;

(b) If v is fully compatible with T, T is a fan if and only if $\overline{T}$ is a fan.

Proof.

(a) We use proposition 23 characterizing a fan. Let $b \in A \backslash I$ such that $\overline{b} \notin -\overline{T}$ we shall show that $\overline{b}$ is $\overline{T}$-*rigid*. T being a fan let $t_1+t_2b \in T+bT \subset T \cup bT$ hence there exist t_3 or t_4 such that $t_1+t_2b=t_3$ or $t_1+t_2b=t_4b$. Going down to k_v we get $\overline{t_1}+\overline{t_2}\overline{b}=\overline{t_3}$ or $\overline{t_1}+\overline{t_2}\overline{b}=\overline{t_4}\overline{b}$ hence $\overline{t_1}+\overline{t_2}\overline{b} \in \overline{T} \cup \overline{b}\overline{T}$, and $\overline{T}$ is a fan.

(b) We use the definition 22 of a fan. Assume v is fully compatible with T and $\overline{T}$ is a fan we have to prove that T is a fan. Let $W \supset T$ be such that $-1 \notin W$, $W^* = W \backslash \{0\}$ is a subgroup of K^* and $[K^* : W^*] = 2$, we have to prove that W is an ordering. We first show that $\overline{W}$ is an ordering. If $-1 = \overline{w}$ for some $w \in W \cap A$, then $-1 = w + m$ for some $m \in I$, so $-w = 1 + m \in 1 + I \subset T \subset W$ hence $-1 \in W$ which is impossible. Since $\overline{T}$ is a fan and $\overline{W^*}$ a subgroup of k_v^* such that $\left[k_v^* : \overline{W^*}\right] = 2$, $\overline{W}$ is an ordering. Form the wedge product $W \wedge \overline{W} = W \cdot \pi^{-1}(\overline{W} \backslash \{0\}) = W \cdot (1+I) \subset W \cdot T \subset W$, since from [L] (p.22) $W \cdot \pi^{-1}(\overline{W} \backslash \{0\}) = W \cdot (1+I)$); then $W \wedge \overline{W} \subset W$ holds, hence $W = W \wedge \overline{W}$ is an ordering.

2.3 Trivialization of fans.

A remarkable result is Bröcker's theorem on trivialization of fans ([Brö]).

Theorem 25 *Let K be a real field and $T \subset K$ be a fan. Then there exists a valuation v, fully compatible with T, such that the pushdown $\overline{T}$ in the residue field k_v is a trivial fan.*

The theorem follows from propositions 26 and 27 below. We use the proof given by Lam ([L], p. 94).

Proposition 26 *Let T be a non-trivial fan in the field K. Then there exists a non-trivial valuation v on K, fully compatible with T.*

The proof of proposition 26 requires three lemmas.

Lemma 1. *Let G be an ordered group (written additively), and H be a subgroup of G. If H does not contain a non-trivial convex subgroup of G, then for any positive element $h \in H$ there exists $g \in G \backslash H$ such that $0 < g < h$.*

Proof of lemma 1. Let $C := \{g \in G \mid \exists n \in \mathbb{N} \quad -nh \leq g \leq nh\}$. C is the convex hull of the subgroup of G generated by h, hence a convex subgroup. Assume there does not exist an element g as in the statement, then for any $g \in G$, $0 \leq g \leq h$ implies $g \in H$. By easy induction on n it follows that for any $n \in \mathbb{N}$, $-nh \leq g \leq nh$ implies $g \in H$. Hence $\{0\} \neq C \subseteq H$, contradicting the assumption that H does not contain a non-trivial convex subgroup of G.

Lemma 2. *Let T be a fan in the field K. Let v_1 be a valuation on K with value group Γ_1; if $v_1(T^*)$ does not contain a non-trivial convex subgroup of Γ_1, then v_1 is fully compatible with T.*

Proof of lemma 2. We claim that the condition:

"for every m in the unique maximal ideal $\mathcal{M}_1$, and for every $t \in U_1 \cap T$, a unit belonging to T, $t + m \in T$ implies that $1 + \mathcal{M}_1 \subset T$"
entails that v_1 is fully compatible with T.

We distinguish two cases:

Case 1. Assume $v_1(m) \notin v_1(T^*)$.

In this case $(T \cdot m) \cap U_1 = \emptyset$; so in particular $m \notin -T$, since $v_1(m) > 0$. Since T is a fan, $t+m \in T + T \cdot m = T \cup T \cdot m$. We have to show that $t+m \in T$. Clearly $t + m \in U_1$ because $v_1(t+m) = 0$ since $v_1(t) = 0$ and $v_1(m) > 0$. Since $(T \cdot m) \cap U_1 = \emptyset$ we get $t + m \notin T \cdot m$ hence $t + m \in T$.

Case 2. Assume $v_1(m) \in v_1(T^*)$.

Apply lemma 1 to $H := v_1(T^*)$. Since $v_1(m)$ is a positive element of H there exists x such that $v_1(x) \notin H$ and $0 < v_1(x) < v_1(m)$. Now let $t + m = t' + m'$ where $t' := t + x$ and $m' = m - x$. From $x \in \mathcal{M}_1$ we get $t' \in U_1$, and since $v_1(m') \notin v_1(T^*)$, case 1 gives $t' \in T$. Finally from $v_1(x) < v_1(m)$ we get $v_1(m') = v_1(m - x) = \min\{v_1(m), v_1(x)\} = v_1(x) \notin v_1(T^*)$. Thus using again case 1, we get $t' + m' \in T$, and hence $t + m \in T$.

Lemma 3. *Let $T \subset K$ be a non trivial fan and $P \in \chi_{/T}$. Let $v_P : K^* \longrightarrow \Gamma$ be the canonical valuation associated with P; then $v_P(T^*) \neq \Gamma$. In particular v_P is not the trivial valuation so every ordering in $\chi_{/T}$ is non archimedean.*

For the proof of this last lemma we refer to Lam [L], corollary 12-11 of lemma 12-10 p. 95.

Proof of proposition 26. Given a non trivial fan $T \subset K$, fix $v_0 : K^* \twoheadrightarrow \Gamma_0$ such that $v_0(T^*) \neq \Gamma_0$ (for instance, take $P \in \chi_{/T}$ and let v_0 be the valuation v_P associated with $A(P)$). Now consider the convex subgroups of Γ_0 contained in $v_0(T^*)$; they form a chain under inclusion. The union of them Δ is the largest convex subgroup contained in $v_0(T^*)$. By quotienting we can coarsen the valuation v_0 into a valuation $v_1 : K^* \twoheadrightarrow \Gamma_1 := \Gamma_0/\Delta$. Then $v_1(T^*)$ cannot contain a non-trivial convex subgroup of Γ_1. Hence, by lemma 2, v_1 is fully compatible with T. Since $[\Gamma_1 : v_1(T^*)] = [\Gamma_0 : v_0(T^*)] > 1$, v_1 is a non trivial valuation.

Proposition 27 *For any preordering T in a field K, the followings are equivalent:*

(1) T is a fan in K.

(2) There exists a valuation v_1 on K, fully compatible with T, such that, with respect to v_1, T pushes down to a trivial fan in the residue field, hence $\left[\overline{K}^* : \overline{T}^*\right] \leq 4$.

Proof of proposition 27.

(2)⇒(1) Trivially if v_1 exists, is fully compatible with T, and pushes down to a trivial fan $\overline{T}$, then T is a fan.

(1)⇒ (2) From the previous proposition we know that there exists a valuation v fully compatible with T, hence $\overline{T}$ is a fan in the residue field k_v.

If $\left[k_v^* : \overline{T}^*\right] \geq 8$, then $\overline{T}$ would be a non-trivial fan, and applying lemma 3 to $\overline{T}$ in k_v we would get a non-trivial valuation on k_v fully compatible with $\overline{T}$. But from proposition 12-3 in [L], k_v has no non-trivial valuation fully compatible with $\overline{T}$. Then just take $v_1 = v$.

For the geometric point of view on fans we refer to [AR] and [ABR].

2.4 Valuation fans (any level) and examples.

From now on preorderings are NOT supposed to be quadratic.

Let us recall the definition of a general preordering. A preordering T in a field K is a subset $T \subseteq K$, satifying:

$T + T \subseteq T$, $T \cdot T \subseteq T$, $0, 1 \in T$, $-1 \notin T$, $T^* = T \backslash \{0\}$ is a subgroup of K^*.

The notion of fans of higher level is analogous, and the trivialization theorem applies to fans of higher level (Becker LNM959)

Definition 28 *(Jacob, [J2]). A preordering T in a field K is a valuation fan if and only if for any $x \notin \pm T$ we have either $1 \pm x \in T$ or $1 \pm x^{-1} \in T$.*

More precisely, a preordering T in K is a valuation fan if and only if $A(T) = \{x \in K \mid \exists r \in \mathbb{Q}\ r \pm x \in T\}$ is a valuation ring with associated valuation v fully compatible with T, and $\overline{T}$ in k_v is an (archimedean) ordering.

There is an alternative characterization for valuation fans:

Proposition 29 *(Jacob, [J1]). Let K be a field; a valuation fan in K is a preordering T such that there exists v a real valuation on K, v fully compatible with T (meaning $1 + I_v \subset T$), and T induces an archimedean ordering on the residue field k_v.*

Example 30 *Usual orderings P are valuation fans (of level 1, i.e. $\sum K^2 \subset P$).*

Example 31 *Valuation fans with no level do exist ; an example is obtained by adding 0 to the set of positive units of $A(P)$*

2.5 Orderings of higher level.

Further examples of valuation fans are provided by Becker's orderings of higher level.

Definition 32 *(Becker, [Be1]). Let K be a commutative real field, $P \subset K$ is an ordering of level n if: $\sum K^{2n} \subset P$, $P + P \subset P$, $P.P \subset P, -1 \notin P$, P^* is a subgroup of K^* and $K^*/P^*$is cyclic.*

When $K^*/P^* \simeq \mathbb{Z}/2n\mathbb{Z}$, then the ordering is said to be of exact level n.

The orderings of level 1 are the usual total orderings.

Example 33 *If* $K = \mathbb{R}((X))$, *there exist two usual orderings:*

$$P_+ = K^2 \cup XK^2 \text{ and } P_- = K^2 \cup -XK^2$$

And for every integer $n \geq 1$ *there exist two orderings of exact level* n*:*

$$P_{n,+} = K^{2n} \cup X^n K^{2n} \text{ and } P_{n,-} = K^{2n} \cup -X^n K^{2n}.$$

All these orderings are associated to the unique $\mathbb{R}$-place of $\mathbb{R}((X))$, and for the associated valuation they all induce the same archimedean ordering in the residue field.

These higher level orderings have important links with sums of powers; we refer the reader to [Be4] and just mention the following important theorems:

Theorem 34 *(Becker, [Be1]). Let* K *be a real field, then:*
$\sum K^{2n} = \cap P_i$ *where* P_i *is an ordering of level dividing* n*.*

Theorem 35 *(Becker, [Be1]). Let* K *be a field, and let* p *be a prime. The followings are equivalent:*
(1) $\sum K^2 \neq \sum K^{2p}$*.*
(2) K *admits an ordering of exact level* p*.*

In the case where the level is a power of 2, Becker's results yields ([Be1]):

Theorem 36 *In a field* K *the followings are equivalent:*
(1) $\forall a \in K \quad a^2 \in \sum K^4$ *;*
(2) Every real valuation on K *has a 2-divisible value group.*
(3) K *does not admit any ordering of exact level* 2*.*

There exist nother approach of higher level orderings with signatures
Usual orderings can be recast in terms of signatures. A signature is a group morphism, $\sigma : K^* \longrightarrow \{\pm 1\}$, with additivily closed kernel; then $P = ker\sigma \cup \{0\}$ is an ordering of K.

The notion of a signature has a higher level analog:

Definition 37 *(Becker, [Be3]). A signature of level* n *on a field* K *is a morphism of abelian groups:*

$$\sigma : K^* \to \mu_{2n}$$

such that the kernel is additively closed, where μ_{2n} *denotes the group of* $2n$*-th roots of 1.*

Clearly if σ is a signature of level n, then $P = ker\sigma \cup \{0\}$ is an ordering of higher level with exact level dividing n.

There even exists a much more general notion of signature involving valuation fans:

Definition 38 *(Schwartz, [S2]). A generalized signature in a field* K *is a morphism of abelian groups,* $\sigma : K^* \to G$, *such that the kernel is a valuation fan.*

3 The space of valuation fans and the real spectrum of the real holomorphy ring

3.1 $\mathbb{R}$-place associated to an ordering.

For a complete presentation of these notions one can refer to [L], or in a more geometrical setting to [Schü1], [Schü2] and [Schü3].

Let K be a real field and P be an ordering on K. Let v denote the valuation associated to the valuation ring $A(P)$. From previous results we know that $(k_v, \overline{P})$ can be uniquely embedded in $(\mathbb{R}, \mathbb{R}^2)$ since $\overline{P}$ is archimedean. Denote this embedding by i and let π be the canonical mapping from K into $k_v \cup \{\infty\}$ (where if $a \notin A(P)$, then $\pi(a) = \infty$).

Definition 39 *The $\mathbb{R}$-place associated to P is $\lambda_P : K \to \mathbb{R} \cup \{\infty\}$ defined by the following commutative diagram:*

$$\begin{array}{ccc} K & \xrightarrow{\lambda_P} & \mathbb{R} \cup \{\infty\} \\ \pi \searrow & & \nearrow i \\ & k_v \cup \{\infty\} & \end{array}$$

Explicitly $\lambda_P(a) = \infty$ when $a \notin A(P)$, and $\lambda_P(a) = \inf\{r \in \mathbb{Q} \mid a \leq_P r\} = \sup\{r \in \mathbb{Q} \mid r \leq_P a\}$ if $a \in A(P)$. In fact it is known that any $\mathbb{R}$-place arises in this way from some ordering P (see [L], 9.1).

3.2 The space of $\mathbb{R}$-places.

The space of $\mathbb{R}$-places of a field K is the set $M(K) = \{\lambda_P \mid P \in \chi(K)\}$, where $\chi(K)$ denotes the space of orderings of K. $M(K)$ is equipped with the coarsest topology making continuous the evaluation mappings defined for every $a \in K$:

$$e_a : M(K) \longrightarrow \mathbb{R} \cup \{\infty\}$$

$$\lambda_P \mapsto \lambda_P(a)$$

Recall that the usual topology on $\chi(K)$ is the Harrison topology generated by the open-closed Harrison sets:

$$\mathcal{H}(a) = \{P \in \chi(K) \mid a \in P\}.$$

With this topology $\chi(K)$ is a compact totally disconnected space. Craven has shown in [C] that every compact totally disconnected space is homeomorphic to the space of orderings $\chi(K)$ of some field K.

Now consider the mapping:
$\Lambda : \chi(K) \longrightarrow M(K)$
$P \mapsto \lambda_P$

With the previous topologies on $\chi(K)$ and $M(K)$ the mapping Λ is a continuous, surjective and closed mapping.

$M(K)$ equipped with the above topology is a compact Hausdorff space. Remark that this topology on $M(K)$ is also the quotient topology inherited from the above topology on $\chi(K)$.

On the side of $\mathbb{R}$-places we know that $\lambda_P = \lambda_Q$ if and only if P and Q are two usual orderings beginning a 2-primary chain of higher level orderings. Such a chain has been defined by Harman in [H].

Hence the mapping $\Lambda : \chi(K) \longrightarrow M(K)$ is a bijection if and only if K does not admit any ordering of exact level 2.

3.3 $\mathbb{R}$-places and the Real Holomorphy Ring.

We now provide some facts on the real holomorphy ring which has heavy links with orderings and $\mathbb{R}$-places.

Definition 40 *The real holomorphy ring, denoted $H(K)$, is the intersection of all real valuation rings of K.*

From the results in part 1 we obtain $H(K) = \bigcap_{P\in\chi(K)} A(P)$.

We also have:

$$H(K) = A(\sum K^2) = \{a \in K \mid \exists n \in \mathbb{N},\ n \geq 1,\ n \pm a \in \sum K^2\}.$$

$H(K)$ is a Prüfer ring with quotient field K (see [L]). Recall that a Prüfer ring is a ring $R \subset K$ such that, for any prime ideal $\mathfrak{p}$ in R, the localization $R_{\mathfrak{p}}$ is a valuation ring in K.

In the sequel we denote the real spectrum of the real holomorphy ring of K:

$$Sper(H(K)) = \{\alpha = (\mathfrak{p},\overline{\alpha}),\ \mathfrak{p} \in Spec(H(K)),\ \overline{\alpha} \text{ ordering of } quot(H(K)/\mathfrak{p})\}.$$

Relations between $\chi(K)$, $M(K)$ and $H(K)$ are given by the next theorem.

Theorem 41 *(Becker-Gondard, [BG2]). The following diagram is commutative:*

$$\begin{array}{ccc} \chi(K) & \xrightarrow{sper\ i} & MinSperH(K) \\ \downarrow \Lambda & & \downarrow sp \\ M(K) \xrightarrow{res} & Hom(H(K),\mathbb{R}) & \xrightarrow{j} MaxSperH(K) \end{array}$$

where the horizontal mappings are homeomorphisms, and the vertical ones continuous surjective mappings (see definitions below).

Hence $\chi(K)$ the space of orderings of K is homeomorphic to $MinSperH(K)$, and the space $M(K)$ of $\mathbb{R}$-places on K is homeomorphic to $MaxSperH(K)$.

The mappings in the above diagram are defined as follows:

$\Lambda : \chi(K) \longrightarrow M(K)$ is given by $P \mapsto \lambda_P$.

$sper\ i : \chi(K) \longrightarrow MinSperH(K)$ is given by $P \mapsto P \cap H(K)$.

$sp : MinSperH(K) \longrightarrow MaxSperH(K)$ is given by $\alpha \longmapsto \alpha^{\max}$,
where $\alpha^{\max}$ is the unique maximal specialization of α.

$res : M(K) \longrightarrow Hom(H(K), \mathbb{R})$ is given by $\lambda \mapsto \lambda_{|H(K)}$.

$j : Hom(H(K), \mathbb{R}) \longrightarrow MaxSperH(K)$ is given by $\varphi \mapsto \alpha_\varphi$,
where $\alpha_\varphi = \varphi^{-1}(\ \mathbb{R}^2)$ or, using the notation for the real spectrum,
$\alpha_\varphi = (\ker \varphi, \overline{\alpha})$ with $\overline{\alpha} = \mathbb{R}^2 \cap quot(\varphi(H(K))$.

All the spaces in the diagram are compact and the topologies of $M(K)$ and $MaxSperH(K)$ are the quotient topologies inherited through Λ and *sp*.

3.4 The space of level 1 valuation fans

From what is said before it is interesting to study in the field case the space of level 1 valuation fans $ValFan(K)$, and its relation with $SperH(K)$.

The motivation is that $\chi(K)$ homeomorphic to $\min SperH(K)$, consists of valuation fans P_i, and that to a $\mathbb{R}$-place λ in $M(K)$, which is homeomorphic to $\max SperH(K)$, can be associated a valuation fan of level 1: $T_\lambda = \cap P_i$ where $P_i \in \Lambda^{-1}(\lambda)$ and $\Lambda^{-1}(\lambda) = \{P_i \mid \lambda_{P_i} = \lambda\}$. We could also work on the same question dealing with signatures.

It is important for real algebraic geometry to understand minimal valuation fans of level 1. They are defined as valuation fans not properly containing any valuation fan which is a quadratic preordering.

Of course such a minimal valuation fan T_0 pushes down an archimedean ordering in the residue field of K for the valuation associated to the valuation ring given by: $A(T_0) = \{x \in K \mid \exists r \in \mathbb{Q}\ r \pm x \in T_0\}$.

But a better way to understand these minimal valuation fans, in relation with $\mathbb{R}$-places, is:

Example 42 *Let λ be a $\mathbb{R}-place$ on a field K, let $\Lambda^{-1}(\lambda) = \{P_i \mid \lambda_{P_i} = \lambda\}$, then $T = \cap P_i$, where P_i belongs to $\Lambda^{-1}(\lambda)$, is a valuation fan and it is a minimal valuation fan of level* 1.

As noticed by Eberhard Becker these valuation fans with no level are also minimal valuation fans.

Given a R-place λ they can be seen as $\lambda^{-1}(\mathbb{R}_+^*)$ Note that $\lambda^{-1}(\mathbb{R}_+^*)$ is also the set E^+ of positive units of $A(\lambda)$ the valuation ring associated to λ. We have $A(\lambda)^* = E^+ \cup -E^+$.

To recover the valuation ring associated to the place λ one could try:

$V_\lambda = \lambda^{-1}(\mathbb{R}_+^*) \cup -\lambda^{-1}(\mathbb{R}_+^*) \cup \{x \in K \mid \forall r \in \mathbb{Q}\ r \pm x \in \lambda^{-1}(\mathbb{R}_+)\}$

Conjecture 43 *(Becker, oral communication)* $\lambda^{-1}(\mathbb{R}_+^*) = \mu^{-1}(\mathbb{R}_+^*)$ *implies* $\lambda = \mu$

For level 1 valuation fans

If T_0 is a level 1 valuation fan contained in P and $\lambda = \lambda_P$ then T_0 contains $K^2\lambda^{-1}(\mathbb{R}_+)$ which is a minimal level 1 valuation fans and we get that

$V_\lambda = A(T_0) = A(P)$.

Similarly for higher level we can consider $K^{2n}\lambda^{-1}(\mathbb{R}_+)$

OPEN PROBLEMS ON $ValFan(K)$ TO BE STUDIED:

1 - Describe the topology on the space $ValFan(K)$ (Harrison or ...).

Note that if K is a totally archimedean field $\chi(K)$ coincides with $ValFan(K)$ hence the topology on $ValFan(K)$ must coincides with the Harrison topology in such cases.

2 - Study, reminding, theorem 41 the relation between $ValFan(K)$ and $SperH(K)$ (recall $H(K)$ is a Prufer ring)

3 - We could perhaps better use the complete real spectrum of $H(K)$ (which is a spectral space), see [GM].

Points in the complete real spectrum are triples (p, v, P) where p is a real prime of $H(K)$, v a real valuation on the residue field $k(p) = qf(H(K)/p)$, and P an ordering of the residue field B/m where B is the valuation ring of v and m its maximal ideal.

4 - Finally is there any relation between the space of valuation fans $ValFan(K)$ and the theory of lattices ?

3.5 More on the space of $\mathbb{R}$-places $M(K)$. and its connected components

Ii is known that: $M(K)$ connected iff $M(K(X))$ connected iff $M(K((X)))$ connected cf. Schülting

Higher level orderings provide a tool to separate connected components in the space of $\mathbb{R}$-places $M(K)$.

Theorem 44 *(Becker-Gondard, [BG2]). Let K be a real field. Two $\mathbb{R}$-places λ_P and λ_Q, associated to usual orderings P and Q, are in two distinct connected components of $M(K)$ if and only if:*

$$\exists b \in K^* \ (b \in P \cap -Q \text{ and } b^2 \in \sum K^4).$$

Proof. This criterion is obtained using higher level orderings, more precisely orderings of exact level 2.

Recall $\mathcal{H}(a) = \{P \in \chi(K) \mid a \in P\}$, $\chi(K) = \mathcal{H}(a) \cup \mathcal{H}(-a)$ and for $a \neq 0$ $\mathcal{H}(a) \cap \mathcal{H}(-a) = \varnothing$, but $\Lambda(\mathcal{H}(a)) \cap \Lambda(\mathcal{H}(-a))$ may be non empty.

Nevertheless, if there exist $b \notin \sum K^2$ with $b^2 \in \sum K^4$, then there does not exist $P \in \mathcal{H}(b)$ and $Q \in \mathcal{H}(-b)$ such that $\lambda_P = \lambda_Q$.

Otherwise $b \notin (P \cap Q) \cup -(P \cap Q)$ and $\lambda_P = \lambda_Q$ imply, as said before, that there exists an ordering of level 2, P_2, such that:

$$P_2 \cup -P_2 = (P \cap Q) \cup -(P \cap Q)$$

with $b \notin P_2 \cup -P_2$, hence $b^2 \notin P_2$, so $b^2 \notin \sum K^4 = \cap P_{2,i}$, where the $P_{2,i}$ run over the set of all orderings with level dividing 2.

Assume that λ_P et λ_Q (with $P \neq Q$) are in the same connected component C of $M(K)$, and that there exists $b \in K^*$ such that $b \in P \cap -Q$ with $b^2 \in \sum K^4$. Λ being closed $C \cap \Lambda(\mathcal{H}(b))$, and $C \cap \Lambda(\mathcal{H}(-b))$ form a partition of C into two non empty closed sets, impossible.

Conversely:

If λ_P et λ_Q are in C and C', two distinct connected components of $M(K)$, $M(K)$ being a compact Hausdorff space there exists an open-closed set U such that $U \supset C$ and $U^c = (M(K) \backslash U) \supset C'$.

Let $X = \Lambda^{-1}(U)$ and $Y = \Lambda^{-1}(U^c)$. X and Y form a partition of $\chi(K)$. Λ being surjective we get : $\Lambda^{-1}(\Lambda(\Lambda^{-1}(U))) = \Lambda^{-1}(U)$ so $\Lambda^{-1}(\Lambda(X)) = X$, and similarly $\Lambda^{-1}(\Lambda(Y)) = Y$.

The following lemma from Harman ensures then the existence of b such that $X = \mathcal{H}(b)$ and $Y = \mathcal{H}(-b)$ with $b^2 \in \sum K^4$, hence we have $b \in P \cap -Q$ with $b^2 \in \sum K^4$.

Harman's Lemma ([H]). If $\chi(K) = \chi_1 \cup \chi_2$, where χ_1 and χ_2 are disjoint open-closed sets such that $\Lambda^{-1}(\Lambda(\chi_1)) = \chi_1$ and $\Lambda^{-1}(\Lambda(\chi_2)) = \chi_2$, then there exist a such that $\chi_1 = \mathcal{H}(a)$ and $\chi_2 = \mathcal{H}(-a)$.

When it is finite, we know the number of connected components of M(K)

Lemma 45 *For any real field K:*

$$\mid \pi_0(M(K)) \mid = 1 + \log_2[(K^{*2} \cap \sum K^4) : (\sum K^{*2})^2].$$

Note that we know from [Be2] page 157 that:
$\sum(K^2)^2 \subset \sum K^4$

Sketch of proof of lemma 45:

It is known from [B2] that $\mid \pi_0(M(K)) \mid = \log_2[E : E^+]$, where E is the group of units of the real holomorphy ring $H(K)$, and $E^+ = E \cap \sum K^2$.

Then we prove that the quotient group $(K^{*2} \cap \sum K^4)/(\sum K^{*2})^2$ is isomorphic to $E/(E^+ \cup -E^+)$.

OPEN QUESTIONS ON THE SPACE OF $\mathbb{R}$-places (field case)

1 - It would be useful to study for a field K the space of connected components of the space of $\mathbb{R}$-places of K: $\pi_0(M(K))$. This must be some kind of space of orderings. It is known from Schulting (see [L] p.79) that $\pi_0(M(K))$ is a compact Hausdorff and totally disconnected space, hence from Craven (Trans AMS 209, 1975) $\pi_0(M(K))$ is a space of orderings.

The question is: which field realizes $\pi_0(M(K))$ as a space of orderings ?

2 - Another question in this area is: in which cases are the connected components of $M(K)$ homeomorphic?

3 - Characterize the topological spaces which are realizable as spaces of $\mathbb{R}$-places. Partial results in that direction have been recently obtained in [EO], [KK], [KMO] and [MMO].

4 Towards abstract spaces of valuation fans.

The space of orderings of a field, studied in relation with quadratic forms and real valuations, have been the origin of the theory of abstract spaces of orderings (1979-80) and of Marshall's problem:

"Is every abstract space of orderings the space of orderings of some field ?"

In [M] it is proved that one can always associate to an abstract space of orderings a "P-structure" (partition of the space of orderings into subspaces which are fans, and such that any fan intersects only one or two classes). Such a P-structure is a candidate to be analogous to the space of $\mathbb{R}$-places in the field case. But it appeared that not every P-structure is a Hausdorff space, hence we have to improve this notion to fit with the space of $\mathbb{R}$-places in the field case.

We need to construct a finer theory for abstract spaces of orderings taking into account the $\mathbb{R}$-places. For example, $\mathbb{Q}(2^{\frac{1}{2}})$ and $\mathbb{R}((X))$ have isomorphic spaces of orderings, but the first one has two $\mathbb{R}$-places and no ordering of level 2, and the second one has only one $\mathbb{R}$-place but has a 2-primary chain of higher level orderings.

We might try to define a notion of abstract space of valuation fans, and deduce a theory of abstract $\mathbb{R}$-places. Both are linked because of the minimal valuation fans of level 1 defined from a $\mathbb{R}$-place λ by $T_\lambda = \cap P_i$ with $P_i \in \Lambda^{-1}(\lambda)$, where $\Lambda^{-1}(\lambda) = \{P_i \mid \lambda_{P_i} = \lambda\}$

In the abstract setting we use the signatures approach for orderings and fans.

On the side of fans, seen as sets of signatures on a field, a four elements fan of level 1 is characterized by: $\sigma_0\sigma_1\sigma_2\sigma_3 = 1$ and it corresponds to the fan seen as a preordering: $T = \bigcap_{i=0}^{3} \ker \sigma_i \cup \{0\}$.

4.1 Abstract valuation fans in level 1 spaces of orderings

Abstract space of orderings have been introduced using signatures by Marshall in [M]:

Definition 46 *An abstract space of orderings is (X, G), where G is a group of exponent 2 (hence abelian), -1 a distinguished element of G, and X a subset of $Hom(G, \{1, -1\})$ such that:*

(1) X is a closed subset of $Hom(G, \{1, -1\})$;

(2) $\forall \sigma \in X \quad \sigma(-1) = -1$;

(3) $\bigcap_{\sigma \in X} \ker \sigma = 1 \quad$ (where $\ker \sigma = \{a \in G \mid \sigma(a) = 1\}$);

(4) For any f and g quadratic forms over G:

$$D_X(f \oplus g) = \cup\{D_X \langle x, y\rangle \mid x \in D_X(f),\ y \in D_X(g)\}.$$

In the above definition $D_X(f)$ denotes the set $\{a \in G$ represented by $f\}$, i.e. there exists g such that $f \equiv_X \langle a\rangle \oplus g$ where $f \equiv_X h$ if and only if f and h have same dimension, and have for any $\sigma \in X$ same signature.

In the abstract case abstract fans have been defined by Marshall.

Definition 47 *An abstract fan is an abstract space of orderings (X, G) such that $X = \{\sigma \in Hom(G, \{1, -1\}) \mid \sigma(-1) = -1\}$.*

It is also characterized by: if $\sigma_0, \sigma_1, \sigma_2 \in X$ then the product $\sigma_0\sigma_1\sigma_2 \in X$.

What was expected to correspond to the space of $\mathbb{R}$-places of the field case in the context of abstract spaces of orderings is called a P-structure and has been defined as follows by Marshall in [M3].

Definition 48 *A P-structure is an equivalence relation on a space of orderings (X, G) such that the canonical mapping $\Lambda : X \to M$, where M is the set of equivalence classes, satisfies:*

(1) Each fiber is a fan;

(2) If $\sigma_0\sigma_1\sigma_2\sigma_3 = 1$ then $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$ has a non empty intersection with at most two fibers.

Marshall has proved that every abstract space of orderings has a P-structure, generally not unique. But unlike the case of the space of $\mathbb{R}$-places in a field, this P-structure M equipped with the quotient topology, is not always Hausdorff.

We know from Craven that SAP abstract spaces of orderings are realizable (see the list of known cases after). Hence we can consider only the non SAP case of abstract spaces of orderings having non trivial fans (ie stability index bigger or equal to 2)

We introduce a new definition :

Definition 49 *An admissible P-structure is a P-structure such that if $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$ is a non trivial 4-elements fan, hence $\sigma_0\sigma_1\sigma_2\sigma_3 = 1$, then any two signatures are in the same fiber of the P-structure, i.e. any two signatures of the fan $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$ are in relation through the equivalence relation used to define the P-structure.*

Equivalently to get a definition of admissible P-structure one could just replace in definition 48 the xiom (2) by:

(2') If $\sigma_0\sigma_1\sigma_2\sigma_3 = 1$ then $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$ has a non empty intersection with a single fiber

The first question to answer is: does an admissible P-structure always have an Hausdorff topology ?

Another option, suggested by Eberhard Becker, is that since Marshall's P-structures are not rich enough to reflect the field case one possible improvement to Marshall's P-structure could be: add one axiom to his definition of P-structure:

(3) M is Hausdorff

Definition 50 *An abstract valuation fan is a fan included in a fiber of an admissible P-structure.*

The idea is that there does not exists a valuation fan in the field case attached to two different $\mathbb{R}$-places, hence in the abstract case an abstract valuation fan cannot have a non empty intersection with two distinct fibers of the P-structure.

Is there also some relation with lattices ? If two valuation fans have a non empty intersection, then there exist a valuation fan containing both (seen as sets of signatures).

Note that single orderings are valuation fans.

4.2 Abstract spaces of signatures (higher level)

In the higher level case, one can also define abstract spaces of signatures.

Definition 51 *An abstract space of signatures of level 2^n is (X, G), G abelian group of exponent 2^n, $X \subset Hom(G, \mu_{2^n})$ such that:*

(0) $\forall \sigma \in X$, $\forall k \in \mathbb{N}$ with k odd, $\sigma^k \in X$;
(1) X is a closed subset of $Hom(G, \mu_{2^n})$;
(2) $\forall \sigma \in X \quad \sigma(-1) = -1$ (-1 distinguished element of μ_{2^n});
(3) $\bigcap_{\sigma \in X} \ker \sigma = 1$ (where $\ker \sigma = \{a \in G \mid \sigma(a) = 1\}$);
(4) For any f and g forms over G

$$D_X(f \oplus g) = \cup\{D_X \langle x, y\rangle \mid x \in D_X(f),\ y \in D_X(g)\}.$$

In fields, the space of $\mathbb{R}$-places is known as soon as one knows the usual orderings and the orderings of level 2. Using this idea in the abstract situation we have been able to obtain a theorem which can be seen as the first case of a P-structure which looks like an abstract space of $\mathbb{R}$-places.

Theorem 52 *(Gondard-Marshall, [GM]). Let (X, G) be a subspace of a space of signatures (X', G') with 2-power exponent.*

For $\sigma_0, \sigma_1 \in X$, define $\sigma_0 \sim \sigma_1$ if $\sigma_0\sigma_1 = \tau^2 \in X'^2$.

Then the followings are equivalent:

(1) If $\sigma_0\sigma_1\sigma_2\sigma_3 = 1$, then either σ_0 is in relation by $\sim$ with exactly one of the $\sigma_1, \sigma_2, \sigma_3$, or σ_0 is in relation by $\sim$ with everyone of the $\sigma_1, \sigma_2, \sigma_3$.

(2) $\sim$ defines a P-structure on X.

Moreover in this case the induced P-structure defined on X by $\sim$ has a Hausdorff topology.

The key idea for proving the theorem is that in the field case, studied by Harman in [H], for any P_2, ordering of level 2, holds for some orderings P_0, P_1:
$a^2 \in P_2 \Longleftrightarrow a \in P_2 \cup -P_2 = (P_0 \cap P_1) \cup (-(P_0 \cap P_1))$.
Hence on the side of abstract signatures we get $\tau(a^2) = \tau(a)^2 = \sigma_0(a)\sigma_1(a)$.
A definition weaker than definition 49 following from the previous example could be :

Definition 53 *A pseudo-admissible P-structure is a P-structure such that if $\sigma_0\sigma_1\sigma_2\sigma_3 = 1$, then either σ_0 is in relation by $\sim$ with exactly one of the $\sigma_1, \sigma_2, \sigma_3$, or σ_0 is in relation by $\sim$ with everyone of the $\sigma_1, \sigma_2, \sigma_3$.*

Note that abstract spaces of orderings can have several P-structure. We only ask that one of these P-structure is admissible. Remind for example that even in the plain case of $\mathbb{Q}(2^{\frac{1}{2}})$ and $\mathbb{R}((X))$, they have isomorphic spaces of orderings but this space admits two different P-structures.

Hence once you have a pseudo-admissible P-structure you can probably construct an admissible P-structure.

With this new definition we keep the definition of abstract valuation fans as fans included in a single fiber of a pseudo-admissible P-structure.

5 Marshall's problem revisited

Definition 54 *The space of signatures* (X,G) *is realizable as the space of orderings of a field when* (X,G) *is isomophic to* $(X_{\sum K^2}, K^*/\sum K^{*2})$.

A more general notion of realizability is as preordered field (K,T) asking (X,G) isomophic to $(X_T, K^*/T^*)$.

Conjecture 55 *An abstract space of orderings is realizable as the space of orderings of a field if (or if and only if?) it is SAP or it admits an admissible P-Structure.*

LIST OF KNOWN CASES FOR MARSHALL'S PROBLEM

There are already spaces of orderings known to be realizable as the space of orderings of a field. So we could first verify that in some of these known cases the (non SAP) space of orderings admits an admissible P-structure.

- SAP spaces of orderings or spaces with stability index less or equal to 1 i.e. there exist only trivial fans - are realizable; these are even realizable as space of orderings of a Pythagorean field (T. Craven, 1975).

- Finite abstract spaces of orderings are realizable (L. Bröcker 1977 and T. Craven 1978).

M. Marshall proved that these spaces of orderings can be realized as the spaces of orderings of Pythagorean fields (1979).

- Spaces of orderings with P-structure M finite (M. Marshall) These are also spaces of orderings with finite chain length (see below *): if M is a $P-structure$ of a space of orderings (X,G) then $cl(X,G) < \infty \Leftrightarrow |M| < \infty$. Moreover if $|M| < \infty$, then $|M| \leq cl(X,G) \leq 2\,|M|$.

- Spaces of orderings with P-structure M with a finite number of realizable connected components (M. Marshall).

- Direct sums and group extension of realizable spaces of orderings (T. Craven for level 1, and V. Powers for higher level).

The *direct sum* of spaces of signatures (X_1, G_1) and (X_2, G_2) is the space of signatures (X,G) where $G = G_1 \times G_2$ and $X = (X_1 \times 1) \cup (1 \times X_2)$.

A space of signatures (X,G) is a *group extension* of a space of signatures (X', G') if G' embeds in G and $X = \left\{\sigma \in \chi\,(G) \;\mid \sigma\restriction_{G'} \in X'\right\}$.

- Direct limits of finite spaces of orderings are realizable (V. Astier et H. Mariano 2011).

- Inverse limit of finite spaces of orderings are realizable (P. Gladki, 2010).

Another question, partially open for some realizable spaces is: *: when a space of orderings is realizable is it always realizable as the space of orderings of a pythagorean field ?*

(*) Some details on chain length

The *chain length of a space of orderings* (X, G), denoted by $cl(X, G)$, is the maximum integer d such that there exists elements $a_0, a_1, ..., a_d \in G$ which satisfy $U(a_0) \subsetneq ... \subsetneq U(a_d)$, or ∞ if such finite d does not exist, where $U(a) := \{x \in X \mid x(a) = 1\}, a \in G$.

- The singleton has chain length 1, and conversely.
- Spaces of orderings which are *fans* F have chain length $cl(F) \leq 2$.
- Any subspace $(Y, G_{|Y})$ of a space of orderings (X, G) satisfies $cl(Y, G_{|Y}) \leq cl(X, G)$.
- The chain length of the direct sum $X_1 \oplus X_2$ satisfies $cl(X_1 \oplus X_2) = cl(X_1) + cl(X_2)$.
- The chain length of (X, G), a group extension of (X', G'), is equal to the chain length of (X', G').

OTHER UNVISITED APPROACHES FOR "Marshall's problem"

- First write an axiomatization for spaces valuation fans (instead of spaces of orderings). Then conjecture : "Is any abstract space of valuation fans the space of valuation fans of some field?"

- Do we consider only level 1 valuation fans ? Or do we consider any level ?

- We could also consider some space corresponding to $Sper H(K)$ or possibly some spectral space corresponding to the complete real spectrum of $H(K)$ denoted $Sper_c H(K)$ and already considered in G-M ?

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