On Valuation Fans in Real Fields

Danielle GONDARD-COZETTE
Université Paris VI,
4, Place Jussieu,
75252 Paris cedex 05 France
gondard@math.jussieu.fr

July 5, 2006

Abstract

The purpose of this course, held during the trimester on "Real Geometry" in CENTRE EMILE BOREL at Institut Henri Poincaré (Paris) during October-November 2005, is to present valuation fans and to study theories corresponding to closure under algebraic extensions of a field equipped with a valuation fan.

On the way we shall present other mathematical objects interesting for Real Algebraic Geometry, like $\mathbb{R}-places$, real holomorphy ring and orderings of higher level, and give some applications.

Some elements of Model Theory for these theories of fields will also be provided.

The frame of the course will be a real field K, which means that -1 is not a finite sum of squares of elements of K. It is well known from Artin-Schreier theory that such a field can be ordered.

1 Compatibility of a valuation with an ordering.

1.1 Preorderings, orderings.

Definition 1 *A preordering T of K is a subset $T \subseteq K$, satifying :*

$$T+T \subseteq T,\ T.T \subseteq T,\ 0,1 \in T,\ \ -1 \notin T \text{ and } T^* = T\backslash\{0\} \text{ is a subgroup of } K^*.$$

The preordering T is called a quadratic preordering when $K^2 \subseteq T$.

Zorn's lemma shows the existence of maximal quadratic preorderings which are orderings.

These are characterized by :

Definition 2 *A subset P of K is an ordering if :*

$$P + P \subseteq P, P.P \subseteq P, P \cup -P = K, -1 \notin P$$

$>$From these properties one can easily deduce that $0, 1 \in P, P \cap -P = \{0\}$ and $\sum K^2 \subseteq P$

We can also call P a positive cone : to any such ordering P one can associate a binary relation, which is a total order relation, as follows :

$$b - a \in P \Leftrightarrow a \leq_P b.$$

Then P is the set of positive elements.

Example 3 *The field $\mathbb{R}$ admits only one ordering with positive cone $P = \mathbb{R}^2$.*

Example 4 *The field $\mathbb{Q}(\sqrt[2]{2}) := \left\{a + b\sqrt[2]{2} \mid a, b \in \mathbb{Q}\right\}$ admits two orderings, one making $\sqrt[2]{2}\,\text{positive}$ and the other making $\sqrt[2]{2}\,\text{negative}$.*

Example 5 $\mathbb{R}((\mathbb{X}))$ *the power series field admits also two orderings making X infinetisimal positive or negative.*

Example 6 $\mathbb{R}(\mathbb{X})$ *admits infinitely many orderings. For any $a \in \mathbb{R}$ one can define $P_{a,+}$ and $P_{a,-}$ making $X - a$ respectively infinitesimal positive or negative.*

1.2 Real Valuations

Definition 7 *A Krull valuation v on a field K is a surjective map*

$$v : K^* \twoheadrightarrow \Gamma$$

where Γ is a totally ordered abelian group, such that
(1) $v(xy) = v(x) + v(y)$ for any x, y in K^*;
(2) $v(x + y) \geq \min\{v(x), v(y)\}$, for any x, y in K^*, with $x + y$ in K^*.

The valuation ring of v is

$$A_v := \{x \in K \mid x = 0 \text{ or } v(x) \geq 0\}$$

and its maximal ideal is

$$I_v := \{x \in K \mid x = 0 \text{ or } v(x) > 0\}$$

$U_v := A_v \setminus I_v$ will denote the group of units and $k_v := A_v/I_v$ the residue field of the valuation.

Definition 8 *Such a valuation v is said to be real if and only if the residue field k_v is formally real (which means $-1 \notin \sum k_v^2$).*

Remark 9 *A field admits real valuations if and only if it is formally real.*

Of course a real field admits real valuations at least the trivial one.

Given a ordering P in a field K, one can define the convex hull of $\mathbb{Q}$:

$$A(P) := \{x \in K \mid \exists r \in \mathbb{Q} \ \ r \pm x \in P\}$$

which is a valuation ring (means that for any $x \in K$ either x or x^{-1}belongs to the ring) and

$$I(P) := \{x \in K \mid \forall r \in \mathbb{Q}^* \ \ r \pm x \in P\}$$

is its unique maximal ideal.

$A(P)$ is clearly a subring of K ; it is a valuation ring because assuming $b \notin A(P)$ we can prove $b^{-1} \in A(P)$: let $b \notin A(P)$, assume $b > 0$, since $b \notin A(P)$ we have in particular $1 < b$,therefore $0 < b^{-1} < 1$ which implies that $b^{-1} \in A(P)$ because $A(P)$ is convex from its definition as convex hull of $\mathbb{Q}$ for P.

We shall see later that this valuation is compatible with the ordering P and pushes down on the residue field an archimedean ordering, hence the valuation is real.

Conversely we shall see below that the well known Baer-Krull theorem ensure that if k_v admits an ordering, then K admits also at least one ordering.

1.3 Compatibility of an ordering with a valuation

Definition 10 *A quadratic preordering T is said to be fully compatible with a valuation v if and only if $1 + I_v \subset T$.*

In this case T induces on the residue field k_v a quadratic preordering $\overline{T}$.

In the special case of an ordering P, we just say that P is *compatible* with v, then $\overline{P}$ induced by P in the residue field k_v is an ordering of k_v.

Example 11 *The trivial valuation, sending every element of K to* 0, *is compatible with any ordering of K.*

Example 12 *The valuation associated to an ordering P with valuation ring*

$$A(P) := \{x \in K \mid \exists r \in \mathbb{Q} \ \ r \pm x \in P\}$$

The valuation associated to a valuation ring A of K, with maximal ideal I, is defined by $v : K^* \to \Gamma$ where $\Gamma := K^*/(A\backslash I)$, and Γ is ordered by $v(x) \leq v(y) \Leftrightarrow yx^{-1} \in A$.

$I(P) := \{x \in K \mid \forall r \in \mathbb{Q}^* \ \ r \pm x \in P\}$ being the maximal ideal we have $1 + I \subset P$ hence the valuation is compatible with P. Then P induces on the corresponding residue field an archimedean ordering $\overline{P}$.

Clearly $\overline{P}$ is closed under addition and multiplication and $\overline{P} \cup -\overline{P} = k_v$; if -1 was in $\overline{P}$ we would have $-1 = \overline{a}$ for some $a \in P \cap A(P)$. Then $1 + a \in I(P)$, hence $-a \in 1 + I(P) \subset P$, so we would get $a = 0$ which is impossible.

Also for any $x \in A(P)$ there exist some $r \in \mathbb{Z}$ such that $-r < x < r$, hence in the residue field we have $-r < \overline{x} < r$ and $\overline{P}$ is an archimedean ordering of k_v.

Theorem 13 *Let P be an ordering of K and v a valuation on K;the following are equivalent :*

(1) $0 <_P a \leq_P b \Rightarrow v(a) \geq v(b)$ in Γ the value group of v.

(2) the valuation ring A_v is convex with respect to P.

(3) the maximal ideal I_v of A_v is convex with respect to P.

(4) v is compatible with P (i.e. $1 + I_v \subset P$).

(1) $\Rightarrow$ (2) A is convex means that $x <_P y <_P z$, with $x, z \in A \Rightarrow y \in A$; equivalently $0 <_P a <_P b$ with $b \in A \Rightarrow a \in A$. From (1) we deduce that $v(a) \geq v(b) \geq 0$ in Γ hence $a \in A$.

(2) $\Rightarrow$ (3) Assume $0 <_P a \leq_P b$ with $b \in A$ then $0 <_P b^{-1} <_P a^{-1}$, but $b^{-1} \notin A$ so using (2) we deduce $a^{-1} \notin A$ hence $a \in I$, I being the ideal of non invertible elements of A.

(3) $\Rightarrow$ (4) Let $m \in I$, if $1 + m \notin P$ then $1 + m \in -P$, so $1 + m <_P 0$ hence $0 <_P 1 < -m$ and using the convexity of I for P,since $-m \in I$ too, this yields $1 \in I$ which is impossible.

(4) $\Rightarrow$ (1) Assume $0 <_P a \leq_P b$ but $v(a) < v(b)$ in Γ ; we deduce $0 < v(b) - v(a) = v(\frac{b}{a})$ hence $\frac{b}{a} \in I$, and also $-\frac{b}{a}$; From (4) we get $1 + (-\frac{b}{a}) \in P$, so $\frac{a-b}{a} > 0$, hence $a > b$ which is impossible.

Theorem 14 ***Proof.***

Theorem 15 *Let $\mathcal{F}$ be the family of all valuation rings compatible with a given ordering P ; then*

(1) The valuation rings in $\mathcal{F}$ form a chain under inclusion.

(2) The smallest element of $\mathcal{F}$ is $A(P)$.

■

(a) Suppose $A, B \in \mathcal{F}$ and $A \not\subseteq B$,let $a \in A \backslash B$ and $a > 0$; we shall prove that $B \subset A$; Consider $0 < b \in B$, from convexity of B we cannot have $0 < a \leq b$ so we must have $0 < b \leq a$, from the convexity of A, we deduce $b \in A$.

(b) Let $A \in \mathcal{F}$, A is convex and contains $\mathbb{Z}$, hence A contains $A(P)$ the convex hull of $\mathbb{Q}$.

Note that any subring of K containing a valuation ring must itself be a valuation ring, hence $\mathcal{F}$ consists of all subrings of K containing $A(P)$; remark also that $A \subset A'$ implies $I' \subset I$.

Definition 16 *The place associated to a valuation ring A, is an application $\lambda : K \to k_v \cup \{\infty\}$, where $\lambda\,|_A$ is the canonical surjection from A to k_v.and which is an homomorphism for sum and product extended to $k_v \cup \{\infty\}$ ($x + \infty = \infty$, and $x\infty = \infty$ with $x \neq 0$). In fact if $a \in A$ then $\lambda(a) = \overline{a} = a + I$ and if $a \notin A$ then $\lambda(a) = \infty$.*

1.4 The Baer-Krull theorem.

Theorem 17 *Let A be a real valuation ring of K. Let $\overline{P}$ be an ordering in the residue field k_v. Let $\chi_{v,\overline{P}}$ be the set of all orderings P_i in K inducing the given $\overline{P}$ in k_v. Then there is a bijection between $\chi_{v,\overline{P}}$ and $Hom(\Gamma, \mathbb{Z}/2)$ where Γ denotes the value group of v.*

The proof needs the following Lemma.

Lemma 18 *Let K be a field and A a real valuation, hence k_v admits at least one ordering with positive cone γ, then there exist at least one ordering on K, with positive cone P compatible with v (or with λ_v the place associated with v) such that $\overline{P}$=γ.*

Let $T := \left\{x \in K \mid \exists y \in K\ \exists z \in A \backslash I\ \lambda_v(z) >_\gamma 0 \text{ and } x = y^2 z\right\}$, we first show that T is a proper quaratic preordering of K.

It is clear that if $x_1, x_2 \in T$ then $x_1 x_2 \in T$ and if $x \in T$ then $x^2 \in T$.

Now suppose that $-1 \in T$ then $\exists y \in K\ \exists z \in A\backslash I$ such that $\lambda_v(z) >_\gamma 0$ and $-1 = y^2 z$, hence $z = -y^{-2}$, but $\lambda_v(-y^{-2}) \leq_\gamma 0$ so we cannot have $\lambda_v(z) >_\gamma 0$; hence $-1 \notin T$.

To show that T is closed under sum, let $x_1, x_2 \in T$, so $x_1 = y_1^2 z_1$ and $x_2 = y_2^2 z_2$ with $z_1, z_2 \in A\backslash I$, $\lambda_v(z_1) >_\gamma 0$ and $\lambda_v(z_2) >_\gamma 0$. Write $x_1 + x_2 = y_1^2 z_1 + y_2^2 z_2 = y_1^2 z_1(1 + z_1^{-1} z_2 y_1^{-2} y_2^2)$, and assume $y_2 y_1^{-1} \in A$ (otherwise $y_1 y_2^{-1}$ is in A). Let $z = 1 + z_1^{-1} z_2 y_1^{-2} y_2^2$, $\lambda_v(z) = 1 + \lambda_v(z_1^{-1} z_2 y_1^{-2} y_2^2) = 1 + (\lambda_v(z_1))^{-1} \lambda_v(z_2)(\lambda_v(y_1^{-1} y_2))^2$ hence $z \in A\backslash I$ with $\lambda_v(z) >_\gamma 0$ and $x_1 + x_2 = (z_1 z) y_1^2$ is in T.

Then there exist P an ordering containing the proper preordering T. A is convex for P because from $1 + I \subset T$ we can deduce $1 + I \subset P$. So suppose $x \in I, v(x) >_\gamma 0$ hence $v(1 + x) = 0$ so $1 + x \in A\backslash I$ and $\lambda_v(1 + x) = 1 >_\gamma 0$. so we can write $1 + x = (1 + x)1^2$ and $1 + x \in T$.

And also we have $\overline{P} = \gamma$ since $P \supset T$ and $\overline{T} = \gamma$ ($\overline{T} \subset \gamma$ is clear, let $\overline{z} \in \gamma$ and z such that $\lambda_v(z) >_\gamma 0$, then $z \in A\backslash I$ and writing $z = z1^2$, one gets $z \in T$.

Now we can give the proof of Baer-Krull theorem.

$>$From the lemma we know that there exist an ordering P, with A convex for P and $\overline{P} = \gamma$. Let Q be any element of χ_K such that Q is compatible with v (or with λ_v) and $\overline{Q} = \gamma$.

Define the following mapping

$\chi_K \longrightarrow Hom(\Gamma, \mathbb{Z}/2)$

$Q \mapsto < P, Q >$

defined by $< P, Q > (v(x)) = 0$ if x has same sign for P and Q, and $< P, Q > (v(x)) = 1$ otherwise.

We first show that $< P, Q >$ is a well defined group homorphism from Γ to $\mathbb{Z}/2$.

It is clear that $x \longmapsto < P, Q > (v(x))$ is a group homomorphism from $K^* \to \mathbb{Z}/2$ with kernel containing $A\backslash I$, because if $x \in A\backslash I$ then $\lambda_v(x) >_\gamma 0$ or $\lambda_v(x) <_\gamma 0$, so for any Q such that $\overline{Q} = \gamma$ we have $x >_Q 0$ or $x <_Q 0$, hence having same sign for P and Q we get $< P, Q > (v(x)) = 0$. Hence $< P, Q >$ is a well defined group homomorphism from Γ to $\mathbb{Z}/2$.

The mapping $Q \mapsto < P, Q >$ is injective because $< P, Q >$ and P entirely define Q (sign of x for Q follows from knowing sign of x for P and $< P, Q > (v(x))$.

We now have to show that the mapping $Q \mapsto < P, Q >$ is surjective. Let $\varphi \in Hom(\Gamma, \mathbb{Z}/2)$. Now define

$Q := \{x \in K \mid x = 0 \text{ or } (\varphi(v(x)) = 0 \text{ and } x \in P) \text{ or } (\varphi(v(x)) = 1 \text{ and } x \in -P\}$

We have to prove that Q is a positive cone of an ordering. It is obvious that $Q \neq K$, $Q.Q \subset Q$, $K^2 \subset Q$, and $Q \cup -Q = K$.

We just prove that $Q + Q \subset Q$. Let $x, y \in Q \setminus \{0\}$, assume $x^{-1}y \in A$ (otherwise $xy^{-1} \in A$), we distinguish two cases.

If $x^{-1}y \in I$, $v(x^{-1}y) > 0$, then $v(1 + x^{-1}y) = 0$, $1 + x^{-1}y \in A \setminus I$ and $1 + x^{-1}y \in P$ because $1 + I \subset P$. Hence $1 + x^{-1}y \in (A \setminus I) \cap P$ which implies that $1 + x^{-1}y \in Q$ since $(A \setminus I) \cap P \subset Q$. Writing $x + y = x(1 + x^{-1}y)$ we get $x + y \in Q$ as product of two elements of Q.

If $x^{-1}y \notin I$ then $x^{-1}y \in A \setminus I$ and $v(x^{-1}y) = 0$ implying $\varphi v(x^{-1}y) = 0$. Since $x^{-1}y \in Q$ we deduce from the definition of Q that $x^{-1}y \in P$. Thus $1 + x^{-1}y \in P$, but also $1 + x^{-1}y \in A$ and since $\lambda_v(1 + x^{-1}y) = \lambda_v(1) + \lambda_v(x^{-1}y)$ we get $\lambda_v(1 + x^{-1}y) > 0$ so $1 + x^{-1}y \notin I$. Finally $1 + x^{-1}y \in (A \setminus I) \cap P$ hence belongs to Q. Again writing $x + y = x(1 + x^{-1}y)$ we get $x + y \in Q$ as product of two elements of Q.

Verify now that A is $Q-convex$: let $m \in I$, $v(m) > 0$ hence $v(1 + m) = 0$, $\lambda_v(1 + m) = 1 >_\gamma 0$, so $1 + m \in P$; $v(1 + m) = 0$ and $1 + m \in P$ imply $1 + m \in Q$.

Also $\overline{Q} = \gamma$ is obvious from $\overline{P} = \gamma$ and definition of Q.

As a consequence, of the Baer-Krull theorem, if $\Gamma/2\Gamma$ has, as vector space over $\mathbb{Z}/2$, a basis of n classes, then $\chi_{v,\overline{P}}$ has 2^n elements P_i ; hence the lifting of $\overline{P}$ to K is unique if and only if Γ is $2-$divisible.

2 Quadratic preorderings and Fans

The compatibility of a preordering with a valuation can be of two kinds. Given T a proper quadratic preordering of a real field K, v is compatible with T if it is *compatible* with some ordering P containing T, and v is *fully compatible* with T if it is compatible with every ordering P containing T. In other words we give the following equivalent definitions.

Definition 19 *Given T a proper quadratic preordering of a real field K and v a valuation of K with unique maximal ideal I :*

(1) v is fully compatible with T if and only if $1 + I \subset T$

(2) v is compatible with T if and only if $(1 + I) \cap -T = \emptyset$, if and only if $\overline{T}$ is a preordering in the residue field k_v.

In the sequence we shall denote by $\chi_{/T} := \{P \text{ } ordering \mid P \supset T\}$.

A way of building fully compatible preorderings is to use the "wedge product" introduced in 1978 by Becker in [ ] and Becker and Brocker in [ ].

Definition 20 *Let K be a real field, A a valuation ring of K, and $\pi : A \longrightarrow k_v$ the projection map ; let T be a preordering of K and S a preordering of k_v such that $S \supset \overline{T}$. The wedge product denoted by $T \wedge S := T.\pi^{-1}(S \backslash \{0\})$.*

We refer the reader to Lam's book [L] p. 21 to verify that $T \wedge S$ is a preordering of K fully compatible with v and such that residually $\overline{T \wedge S} = S$. There is also an alternative definition for the wedge product :

$T \wedge S = \cap \left\{ \text{orderings } P \supset T \mid \overline{P} \in \chi_{/S} \right\}$

2.1 Fans

Fans in the context of preorderings have been presented first by Becker and Kopping in [ ].

Definition 21 *Let K be a real field and T a proper quadratic preordering of K. T is a fan if and only if for any $S \supset T$ with $-1 \notin S$ and $S^* = S \backslash \{0\}$ is a subgroup of K^* such that $[K^* : S^*] = 2$ then S is an ordering of K.*

Note that if T is a fan any preordering containing T is again a fan. There is an alternative useful definition of a fan given in [L] p. 40, with proof of equivalence of definitions.

Definition 22 *A preordering T is a fan if and only if for any $a \in K^* \backslash -T$ we have $T + aT \subset T \cup aT$. (such an element a is said $T-rigid$).*

Examples of fans are provided by what is called trivial fans : orderings P and intersection of two ordeings $P_1 \cap P_2$. Further examples will be given later with orderings of higher level.

Another example is the pullback $\widehat{S}$ of a trivial fan S in k_v, namely $\widehat{S} = K^2 \wedge S = K^2.\pi^{-1}(S^*)$ is a fan in K. In fact the trivialization theorem of Bröcker given later says that all fans arise in this way.

It is important to understand minimal fans, defined as T_0 such that for any quadratic preordering $T_1 \subset T_0$, T_1 cannot be a fan. Such a minimal fan T_0 is the pullback of a trivial fan with respect to the valuation associated to the valuation ring (convex hull of $\mathbb{Q}$ with respect to T_0) given by $A(T_0) = \{x \in K \mid \exists r \in \mathbb{Q} \ r \pm x \in T_0\}$.

Theorem 23 *Let K be a real field, v a valuation, and T a preordering; then the following hold :*

(a) if v is compatible with T, T is a fan implies that $\overline{T}$ is a fan

(b) if v is fully compatible with T, T is a fan if and only if $\overline{T}$ is a fan

Proof of (a) using second definition of a fan. Let $b \in A \backslash I$ such that $\overline{b} \notin -\overline{T}$ we shall show that $\overline{b}$ is $\overline{T} - rigid$. T being a fan let $t_1 + t_2 b \in T + bT \subset T \cup bT$

hence there exist t_3 or t_4 such that $t_1 + t_2b = t_3$ or $t_1 + t_2b = t_4b$. Going down to k_v we get $\overline{t_1} + \overline{t_2}\overline{b} = \overline{t_3}$ or $\overline{t_1} + \overline{t_2}\overline{b} = \overline{t_4}\overline{b}$ hence $\overline{t_1} + \overline{t_2}\overline{b} \in \overline{T} \cup \overline{bT}$, and $\overline{T}$ is a fan.

Proof of (b) using first definition of a fan. Assume v is fully compatible with T and $\overline{T}$ is a fan we have to prove that T is a fan. Let $W \supset T$ be such that $-1 \notin W$, $W^* = W \backslash \{0\}$ is a subgroup of K^* and $[K^* : W^*] = 2$, we have to prove that W is an ordering. We first show that $\overline{W}$ is an ordering ; if $-1 = \overline{w}$ for some $w \in W \cap A$, then $-1 = w + m$ for some $m \in I$, so $-w = 1 + m \in 1 + I \subset T \subset W$ hence $-1 \in W$ which is impossible. Since $\overline{T}$ is a fan and $\overline{W^*}$ a subgroup of k_v^* such that $\left[k_v^* : \overline{W^*}\right] = 2$, $\overline{W}$ is an ordering. Form the wedge product $W \wedge \overline{W} = W.\pi^{-1}(\overline{W} \backslash \{0\}) = W(1 + I) \subset W.T \subset W$ ($W.\pi^{-1}(\overline{W} \backslash \{0\}) = W(1+I)$ from [L] p.22) then $W \wedge \overline{W} \subset W$, hence $W = W \wedge \overline{W}$ is an ordering.

2.2 Trivialization of fans

A very important theorem is the Brocker's theorem of trivialization of fans.

Theorem 24 *Let K be a real field and T be a fan. Then there exist a valuation v fully compatible with T such that the pushdown $\overline{T}$ in the residue field k_v is a trivial fan.*

we refer the reader to Lam ???
or write down the proof ???

3 $\mathbb{R}-$places

3.1 $\mathbb{R}-$place associée à un ordre

For a complete presentation of these notions one can refer to [L] and [Schü]

Let K be a real field and P an ordering on K; from previous results we know that $(k_v, \overline{P})$ can be uniquely embedded in $(\mathbb{R}, \mathbb{R}^2)$ since $\overline{P}$ is archimedean. Denote i this embedding and π the canonical application from K into $k_v \cup \{\infty\}$ (where if $a \notin A(P)$, then $\pi(a) = \infty$).

Definition 25 *The $\mathbb{R}-place$ associated to P is $\lambda_P : K \to \mathbb{R} \cup \{\infty\}$ defined by the following commutative diagram :*

$$\begin{array}{ccc} K & \xrightarrow{\lambda_P} & \mathbb{R} \cup \{\infty\} \\ \pi \searrow & & \nearrow i \\ & k_v \cup \{\infty\} & \end{array}$$

Explicitely $\lambda_P(a) = \infty$ when $a \notin A(P)$, and if $a \in A(P)$, then $\lambda_P(a) = \inf\{r \in \mathbb{Q} \mid a \leq_P r\} = \sup\{r \in \mathbb{Q} \mid r \leq_P a\}$.

3.2 Space of $\mathbb{R}-$places

Definition 26 *The space of $\mathbb{R}$-places of a field is $M(K) = \{\lambda_P \mid P \in \chi(K)\}$, where $\chi(K)$ denotes the space of orderings of K.*

$M(K)$ is equipped with the coarsest topology making continuous the evaluation mappings defined for every $a \in K$ by

$$e_a : M(K) \longrightarrow \mathbb{R} \cup \{\infty\}$$

$$\lambda \mapsto \lambda(a)$$

$M(K)$ with this topology is a compact Hausdorff space and the mapping

$$\Lambda : \chi(K) \ \longrightarrow M(K)$$

$$P \mapsto \lambda_P$$

is continuous, surjective and closed.

Definition 27 *The usual topology on $\chi(K)$ is the Harrison topology generated by the open-closed Harrison sets :*

$$H(a) = P \in \chi(K) \mid a \in P\}.$$

With this topology χ (K) is a compact totally disconnected space.

Craven has shown in [C] that every compact totally disconnected space is homeomorphic to the space of orderings $\chi(K)$ of some field K.

Remark that the topology of $M(K)$ is also the quotient topology inherited from $\chi(K)$.

4 Orderings of higher level

>From now on preorderings will not be supposed to be quadratic. Hence the general definition of a preordering will be :

A preordering T of K is a subset $T \subseteq K$, satifying :

$T + T \subseteq T,\ T.T \subseteq T,\ 0, 1 \in T,\ \ -1 \notin T$ and $T^* = T \backslash \{0\}$ is a subgroup of K^*.

First examples are provided by the Becker's orderings of higher level, which are only partial orders.

Definition 28 *(Becker [Be1]) : Let K be a commutative formally real field, $P \subset K$ is an ordering of level n if : $\sum K^{2n} \subset P$, $P + P \subset P$, $P.P \subset P$ (hence P^* is a subgroup of K^*); and if $K^*/P^* \simeq \mathbb{Z}/2n\mathbb{Z}$ the ordering is of exact level n.*

The notation $\sum K^{2n}$ denotes the set of all finite sums of 2n-th powers of elements of K.

The level 1 orderings are the total usual orders.

Theorem 29 *(Becker [Be1]) :* $\sum K^{2n} = \bigcap_{\text{level of } P \text{ divides } n} P$

Theorem 30 *(Becker [Be1]) : Let p be a prime, $\sum K^2 \neq \sum K^{2p}$ if and only if K admits orderings of exact level p.*

There exists another approach for such objects with higher level signatures.

Definition 31 *(Becker [Be3]) : A signature of level n is a morphism of abelian groups*

$$\sigma : K^* \to \mu_{2n}$$

where μ_{2n} denotes the group of $2n$-th roots of 1 and such that the kernel is additively closed.

Clearly if σ is a signature of level n, then $P = \ker\sigma \cup \{0\}$ is an ordering of higher level, and its level divides n.

Example 32 *Let $K = R((X))$, K admits two usual orders namely*

$$P_+ = K^2 \cup XK^2 \text{ and } P_- = K^2 \cup -XK^2$$

and for any prime p there exist two orderings of exact level p :

$$P_{p,+} = K^{2p} \cup X^pK^{2p} \text{ and } P_{p,-} = K^{2p} \cup -X^pK^{2p}$$

These higher level orderings have important links with sums of powers.

In the sequence $\sum K^{2p}$ will denote the set of finite sums of $2p$-th powers of elements of K.

Theorem 33 *The following are equivalent (p prime) :*
(1) $\sum K^2 \neq \sum K^{2p}$
(2) K admits an ordering of exact level p.

Theorem 34 *For any integer $n \geq 1$ holds*

$$\sum K^{2n} = \bigcap_{P \text{ ordering whose level divides } n} P$$

Example 35 *If $K = \mathbb{R}((\mathbb{X}))$, there exist two usual oredrings*

$$P_+ = K^2 \cup XK^2 \text{ et } P_- = K^2 \cup -XK^2$$

and for every prime p premier there exist two oredrings of exact level p :

$$P_{p,+} = K^{2p} \cup X^pK^{2p} \text{ et } P_{p,-} = K^{2p} \cup -X^pK^{2p}$$

All these orderings are associated to the unique $\mathbb{R}-$place of $\mathbb{R}((X))$, and for the associated valuation they all induce on the residue field the same archimedean ordering.

>From Becker's works one can deduce :

Theorem 36 *The following are equivalent :*
(1) $\forall a \in K \quad a^2 \in \sum K^4$ *;*
(2) every real valuation of K *has a 2-divisible value group;*
(3) K *does not admit any ordering of exact level* 2*.*

As a corollary we obtain that $\lambda_P = \lambda_Q$ if and only if P and Q are the beginning of a 2-primary chain of higher level orderings (such a chain has been defined by Harman [H] as $(P_n) = (P_0, P_1, ..., P_n, ...)$, P_0 usual ordering, P_n ordering of exact level 2^{n-1} satifying the condition $P_n \cup -P_n = (P_0 \cap P_{n-1}) \cup -(P_0 \cap P_{n-1})$)

The mapping $\Lambda : \chi(K) \longrightarrow M(K)$ is a bijection if and only if K does not admit any ordering of exact level 2.

5 Valuation fans

Definition 37 *(Jacob [J1]) : a valuation fan is a preordering* T *such that there exists a real valuation* v*, compatible with* T*, (meaning* $1 + I_v \subset T$*), inducing an archimedean ordering on the residue field* k_v*.*

More precisely a preordering T is a valuation fan if and only if $A(T) = \{x \in K \mid \exists r \in \mathbb{Q} \ r \pm x \in T\}$ is a valuation ring with associated valuation v fully compatible with T, and $\overline{T}$ in k_v is an (archimedean) ordering.

Example 38 *Usual orderings* P *are valuation fans (of level* 1*, i.e.* $\sum K^2 \subset P$*).*

Example 39 P_n *orderings of exact level* n *are valuation fans of level* n*.*

Example 40 *Let* $\Lambda^{-1}(\lambda) = \{P_i \mid \lambda_{P_i} = \lambda\}$ *(where* Λ *is the mapping* : $\chi(K) \longrightarrow M(K)$ *defined by* $P \mapsto \lambda_P$*), then* $T = \cap P_i$ *is a valuation fan and it is a minimal level 1 valuation fan.*

Definition 41 *(Schwartz [S2]) : a generalized signature is a morphism of abelian groups,* $\sigma : K^* \to G$*, such that the kernel is a valuation fan*

Example 42 ***Example 43*** *If* σ *is a group morphism,* $\sigma : K^* \longrightarrow \{\pm 1\}$ *, with kernel additivily closed, then* σ *is a signature, and* $P = ker\sigma \cup \{0\}$ *is an ordering.*

Example 44 *If* $\sigma : K^* \to \mu_{2n}$ *is a morphismof abelian groups, with additively closed kernel, then* $P = \ker \sigma \cup \{0\}$ *is an ordering whose exact level divides* n*.*

6 Algebraic closure of a field equipped with a valuation fan

Many notions of real closure, under algebraic extensions, of a field equipped with either higher level orderings or higher level signatures, either valuation fans or generalized signatures have been introduced and studied.

All these attempts of closures can be unified in one theory, the theory of Henselian Residually Real-Closed Fields (HRRC fields)

Definition 45 *(Becker, Berr, Gondard [BBG]) : A field K is henselian residually real-closed (HRRC) if and only if it admits an henselian valuation v with real-closed residue field k_v.*

Recall that a valuation $v: K^* \twoheadrightarrow \Gamma_v$ is *henselian* if it satisfies Hensel's lemma : *"for any monic polynomial $f \in A_v[X]$, if $\overline{f}$ has a simple root $\beta \in k_v$, then f has a root $b \in A_v$ such that $\overline{b} = \beta$ "*

In the litterature there exist other names for the same theory : the henselian residually real-closed fields are called real henselian fields in Brown [Br], real-closed with respect to a signature in Schwartz [S2] and almost real-closed fields in Delon-Farre [DF].

6.1 Examples of HRRC fields

The basic examples of henselian residually real-closed fields are constructed as follows. Let $K = R((\Gamma))$ where $R((\Gamma)) = \{\sum\limits_{\gamma} a_\gamma t^\gamma \mid \gamma \in \Gamma, a_\gamma \in R\}$ is the set of generalized power series, with support of $\sum\limits_{\gamma} a_\gamma t^\gamma$ well ordered, R a real-closed field and Γ a totally ordered abelian group. In $K = R((\Gamma))$ one can define :

- product by $t^\gamma t^\delta = t^{\gamma+\delta}$;
- sum by $\sum\limits_{\gamma} a_\gamma t^\gamma + \sum\limits_{\delta} b_\delta t^\delta = \sum\limits_{\alpha} (a_\alpha + b_\alpha) t^\alpha$;
- order by $\sum\limits_{\gamma} a_\gamma t^\gamma >_K 0 \Leftrightarrow a_m >_R 0$
where $m = \min(\text{support} \sum\limits_{\gamma} a_\gamma t^\gamma)$;
- valuation as $v: R((\Gamma)) \to \Gamma$ defined by $v(\sum\limits_{\gamma} a_\gamma t^\gamma) = m = \min(\textit{support} \sum\limits_{\gamma} a_\gamma t^\gamma)$.

Then it is known that $R((\Gamma))$ is a field, admitting v as henselian valuation with real-closed residue field R and value group Γ, hence $R((\Gamma))$ is an HRRC field.

6.2 Subtheories of the theory of HRRC fields

Let v be a real valuation on a field K, k_v its residue field and Γ_v its value group. Let S be a set of primes.

Relations between theories are described by the following diagram where arrows map to subtheories.

Henselian Residually Real-Closed Fields (HRRC)

v henselian valuation , k_v real-closed field

closed for **generalized signature**, or for **valuation fan**, or for $\mathbb{R}-$**place**

$\downarrow$

HRRC field of type S ($p \notin S \Rightarrow \Gamma_v$ p-divisible)

$\swarrow \quad \searrow$

<u>*S−generalized real-closed field* (</u>S finite)
if $p \notin S$ then Γ_v is p-divisible
if $p \in S$ and Γ_v is not p-divisible then
$\Gamma_v/p\Gamma_v \simeq \mathbb{Z}/p\mathbb{Z}$; closed for
higher level ordering or **chain signature**

<u>*Rolle field*</u>
Γ_v odd divisible
HRRC field of type $\{2\}$

$\downarrow \quad \searrow \quad \swarrow \quad \downarrow$

<u>*Real-closed field*</u>
$\emptyset$-generalized real-closed
Γ_v divisible
closed for a **usual order**

<u>*Chain-closed field*</u>
$\{2\}$-generalized real-closed
$\Gamma_v/2\Gamma_v \simeq \mathbb{Z}/2\mathbb{Z}$
closed for an **ordering of level** 2^k

As said in the diagram above, any of these theories corresponds to a notion of closure, under algebraic extensions, of a field equipped with some object: with an *order* (real-closed field), with an *ordering of two power exact level* (chain-closed field), with an o*rdering of* $p-$*power exact level where* p *is prime* ($\{p\}$ −real-closed fields), with an *ordering of exact level* n (generalized real-closed fields of exact type S ($p \in S \Leftrightarrow \Gamma_v$ not divisible) and for all $p \in S$, $p \mid n$ with S finite), or with a *valuation fan* (henselian residually real-closed field).

Note that for HHRC fields, an alternative object is a $\mathbb{R}-$*place* (defined part 3), because to any $\mathbb{R}-$ *place* of a formally real field it is possible to associate a canonical level 1 valuation fan in the field . Hence we can consider closures of a field equipped with a $\mathbb{R}-$*place*. These special closures must play some important role in Real Algebraic Geometry (see [BG2] and [G4]).

6.3 On the question of uniqueness of closure

For a field equipped with a true usual order it is well known that the real closure is unique up to $K-$ *isomorphism.*

Even for chain-closed fields this is no more true.

In order to recover such a uniqueness for chain-closed fields one needs to consider a closure for a whole chain of two power levels orderings in the sense of Harman :

Definition 46 *(Harman [H]) : a 2-primary chain of orderings is*

$$(P_n)_{n\in\mathbb{N}} = (P_0, P_1, ..., P_n, ...)$$

P_0 *being a usual order,* P_n *an ordering of level* 2^{n-1}*, such that*

$$P_n \cup - P_n = (P_0 \cap P_{n-1}) \cup -(P_0 \cap P_{n-1})$$

Theorem 47 *(Harman [H]) : a field* K*, with a two power level chain of orderings, admits a closure under algebraic extensions unique up to* $K-$ *isomorphism. The closure is called a chain-closed field and it is equal to the intersection of the two real-closures of* K *for* P_1 *and* P_2*.*

For generalized real-closed fields, Niels Schwartz has introduced the notion of chain signature in order to recover the uniqueness up to K-isomorphism.

Definition 48 *(Schwartz, [S1]) : A chain signature is a homomorphism*

$$\varphi : K^* \to \{1, -1\} \times \widehat{\mathbb{Z}}$$

such that kerφ is a valuation fan, with $\widehat{\mathbb{Z}} = \prod \widehat{\mathbb{Z}}_p$ where $\widehat{\mathbb{Z}}_p$ denotes the additive group of p-adic integers,

One can recover orderings of higher level by taking :
$P_n(\varphi) = \varphi^{-1}(1 \times n\widehat{\mathbb{Z}})$

Theorem 49 *(Schwartz, [S1]) : A field K equipped with a chain signature φ admits a closure under algebraic extensions unique up to $K-$ isomorphism. This closure is a HHRC field.*

In the more general situation of a field equipped with a valuation fan we can also ensure the uniqueness of the closure by considering a field equipped, not only with a single valuation fan, but with a whole chain of valuation fans.
>From Brown's work we can derive the following :

Theorem 50 *(Brown [Br]) : Let R and R' be two algebraic HRRC fields, extensions of a field K, then the followings are equivalent:*
(1) R and R' are $K-$isomophic.
(2) $R^{2n} \cap K = R'^{2n} \cap K$ for all $n \in \mathbb{N}$.

In fact these $T_n = R^{2n} \cap K$ are valuation fans, which form a chain of valuation fans $(T_n)_{n\in\mathbb{N}}$ as defined below; this chain is said to be induced by R.

Definition 51 *(Becker-Berr-Gondard [BBG]) : A chain of valuation fans is defined as $(T_n)_{n\in\mathbb{N}}$ such that:*
(1) $K^{2n} \subset T_n$
(2) $T_{n.m} \subset T_n$
(3) $(T_n)^m \subset T_{n.m}$
(4) $T_n^*/T_{n.m}^* \subset T_1^*/T_{n.m}^*$ is the subgroup of elements of exponent m.

Theorem 52 *(Becker-Berr-Gondard [BBG]) : Any field K, equipped with a chain of valuation fans $(T_n)_{n\in\mathbb{N}}$, admits a closure R, under algebraic extensions, unique up to $K-$isomorphism ; then R is a HRRC field and R induces $(T_n)_{n\in\mathbb{N}}$ (i.e. $T_n = R^{2n} \cap K$ for all n).*

6.4 Properties of HHRC fields

Henselian residually real-closed fields have a lot of nice properties ; we list without proof some of them below.

Let R be an HRRC field then :

(1) R is a real field;

(2) Every algebraic extension of R is a radical extension;

(3) R has no real extension of degree $p \in \mathbb{P}\backslash S$.
Note that whenever $2 \in S$, one can replace (3) by (3') "R has no extension of degree $p \in \mathbb{P}\backslash S$ ";

(4) $\forall n \in \mathbb{N}$, K is pythagorean $K^{2n} + K^{2n} = K^{2n}$;

(5) K is hereditarily pythagorean, which means that every algebraic extension is again a pythagorean field;

(6) $\forall n \in \mathbb{N}$, K^{2n} is a fan :
$0, 1 \in K^{2n}$, $-1 \notin K^{2n}$, $K^{2n} + K^{2n} = K^{2n}$,
K^{2n*} is a subgroup of K^*,
$\forall x \notin -K^{2n}$ holds $K^{2n} + xK^{2n} = K^{2n} \cup xK^{2n}$;

(7) $\forall n \in \mathbb{N}$, K^{2n} is a valuation fan, i.e. it is a preordering such that :
$\forall x \notin \pm K^{2n}$ holds $1 \pm x \in K^{2n}$ or $1 \pm x^{-1} \in K^{2n}$;

(8) all real valuations of K are henselian;

(9) The set of real valuation rings is totally ordered by inclusion;

(10) The *smallest* real valuation ring is :

$$A(K^2) = A(K^{2n}) = H(K)$$

where $A(T) = \{x \in K \mid \exists n \in \mathbb{N} \ \ n \pm x \in T\}$, with T a valuation fan, and where $H(K)$ is the real holomorphy ring (equal to the intersection of all real valuation rings);

(11) This ring $A(K^2)$ is associated to a valuation v corresponding to the unique $\mathbb{R} -$ *place* of K;

(12) The Jacob's ring $J(\cap K^{2n})$ is the *biggest* valuation ring with real-closed residue field. This ring is defined as follows : if T is a valuation fan, the ring $J(T)$ is equal to $J_1(T) \cup J_2(T)$ with

$$\begin{aligned} J_1(T) &= \{x \in K \mid x \notin \pm T \text{ et } 1 + x \in T\} \\ \text{et } J_2(T) &= \{x \in K \mid x \in \pm T \text{ et } xJ_1(T) \subset J_1(T)\}) \end{aligned}$$

The importance of this Jacob's ring will appear later with the transfer theorem obtained by Delon and Farré.

7 Some Model Theory for HHRC Fields and Applications

We make a review of the model theoretic results obtained for the theory of HRRC fields and its subtheories.

7.1 Axiomatization for Rolle fields

A Rolle field is an ordered field where Rolle theorem holds for polynomials. Below is an axiomatization for the theory of Rolle fields; these axioms are first order in the language of fields, hence the theory is elementary.

Theorem 53 *(Gondard [G1]) :*
(1) axioms for commutative fields ;
(2) "K formally real " :
for each $n \geqslant 1$

$$\forall x_1 ... \forall x_n \urcorner(-1 = x_1^2 + ... + x_n^2)$$

(3) "K does not have any algebraic extension of odd degree" :
for each $p \geqslant 0$

$$\begin{array}{rcl} & & \forall x_0 ... \forall x_{2p+1} \exists y \\ (x_{2p+1} & = & 0 \vee x_0 + x_1 y + ... + x_{2p+1} y^{2p+1} = 0) \end{array}$$

(4) "K^2 is a fan " :

$$\forall x \forall y \forall z \exists t (x = -t^2 \vee y^2 + xz^2 = t^2 \vee y^2 + xz^2 = xt^2)$$

(5)"K is pythagorean at level 2 " :

$$\forall x \forall y \exists z (x^4 + y^4 = z^4)$$

The three first sets of axioms are the same as in the theory of real-closed fields, to get a real-closed field axiomatization, just replace (4) and (5) by

$$\forall x \exists y (x = y^2 \vee x = -y^2)$$

In [G1] it is also shown that :

Theorem 54 *For any Rolle field K having a finite number 2^n of orders, there exists $n+1$ orders P_i, such that K is the is the intersection of the $n+1$ real closures R_i of K ordered by P_i.*

7.2 Elementary theory of HRRC fields

Theorem 55 *(Becker, Berr, Gondard [BBG]) : The class of HRRC fields admits the following axiomatization :*
(1) R is a commutative field ;
(2) R is a hereditarily pythagorean field ;
(3) for all $n \in \mathbb{N}$, R^{2n} is a fan.

Corollary 56 *The class of HRRC fields is an elementary class.*

Remark 57 *The class of HRRC fields of type S is also an elementary class, just add to the axiomatization in theorem 18 :*
(4) for all $p \in \mathbb{P} \backslash S$, $K^2 = K^{2p}$.

Corollary 54 follows from B. Jacob ([J1]), who first proved that the class of hereditarily pythagorean fields is elementary.

An alternative proof from [BBG] for "the class of hereditarily-pythagorean fields is elementary" is given below. It uses the characterization by Becker ([B1], thm. 4, p. 94) of hereditarily pythagorean fields :

$$\sum K(X)^2 = K(X)^2 + K(X)^2$$

which is equivalent to :

$$\sum K[X]^2 \subset K(X)^2 + K(X)^2$$

By Cassel's theorem this is also equivalent to :

$$\sum K[X]^2 = K[X]^2 + K[X]^2 \quad (*)$$

Remark that if $f, g, h \in K[X]$ satisfy $f^2 = g^2 + h^2$, the degrees of g and h are less or equal to the degree of f because K is formally real.

Hence $(*)$ is expressible by an infinite sequence of first order sentences in the language of fields.

7.3 A tranfer theorem

Theorem 58 *(Delon-Farré, [DF]) : Let K and L be a HRRC fields, then :*
(i) $K \equiv L \Leftrightarrow \Gamma_{J(K)} \equiv \Gamma_{J(L)}$;
(ii) if $K \subset L$ then
$K \prec L \Leftrightarrow \Gamma_{J(L)}$ extends $\Gamma_{J(K)}$, and $\Gamma_{J(K)} \prec \Gamma_{J(L)}$, where the Γ's are the value groups of the Jacob rings of K and L.

In Delon-Farre it is also proved that the theory of HRRC fields is elementary, and the authors established a bijection between theories of HHRC fields and certain theories of ordered abelian groups. This bijection preserves completeness and sometimes decidability. Finally they proved that the only model-complete theory among these is that of real-closed fields.

They also characterized definable real valuation rings in such fields and have shown that they were in bijection with the definable convex subgroups of the value group of the Becker ring.

In case we have only one real (henselian) valuation ring with real-closed residue field, i.e. the Becker ring equals the Jacob ring, then the model theory works well, and we are able to get real algebraic results such as a Nullstellensatz or a Hilbert's 17th problem at level n.

7.4 Some Applications in HRRC fields

We shall now give examples of applications of the previous parts in some of the theories studied. The proofs often make use of the model theoretic results given above. Further results are mentionned in the bibliography.

Going back to the diagram showing relations between theories, we can list some results related to real algebraic geometry.

HRRC fields
Journal of Algebra, 215, 1999
[BBG]
closure for valuation fan chain

$\downarrow$

HRRC fields of type S
[BBG]
strong Hensel lemma

$\swarrow \quad \searrow$

S-generalized real-closed fields
Journal de Crelle 1994
[BBDG]
Hilbert's 17th pb at level n

$\downarrow \quad \searrow$

Real closed field

Rolle fields
Manuscripta Math. 1990
[G1]
intersection of real closures

$\swarrow \quad \downarrow$

Chain-closed fields
Manuscripta Mathematica 1989
[BG]
Nullstellensatz.

Closures for a field with a valuation fan chain has been studied before in part 6, and Rolle fields as intersection of real closures in part 7. The strong Hensel's lemma, which allow sometimes for to lift multiple roots from the residue field is too technical to be given in few words and we refer the reader to the original paper. We now present below two applications to Real Algebraic Geometry.

7.4.1 Hilbert's 17th problem at level n

Let K be a formally real field, V an irréductible affine variety over K, and $K(V)$ its function field.

In [BBDG] we have searched, depending on K and V, for which $n \in \mathbb{N}$ holds

- strong property Q_n : $\forall f \in K[V]$

$$(f \in \sum K(V)^{2n} \Longleftrightarrow \forall x \in V_{reg}(K)\ f(x) \in \sum K^{2n})$$

- weak property $Q_n^{'}$: $\forall f \in K[V)]$

$$(f \in \sum K(V)^{2n} \Longleftrightarrow \forall x \in V_{reg}(\overline{K})\ f(x) \in \sum K(x)^{2n})$$

We got for instance :

Theorem 59 *(Becker, Berr, Delon, Gondard [BBDG]) :*
$R((G))$, where $G = \{\frac{r}{s} \mid r, s \in Z$ and $p \nmid s\}$, is a $\{p\}$-generalized real-closed field with only one henselian valuation with real-closed residue field. For any variety V, property $Q_n^{'}$ holds for $R((G))$ if and only if $n \in < p >$ the multiplicative semi-group with 1.

7.4.2 Nullstellensatz for chain-closed fields

Theorem 60 *(Becker, Gondard [BG]) : let K be a chain-closed field with only one henselian valuation with real-closed residue field. Let $\alpha \in K$ be such that $\alpha^2 \notin K^4$, then for any ideal $\mathcal{P} \in K[\overline{X}]$holds :*

$$\begin{array}{rcl} I_K(V_K(\mathcal{P})) & = & \{f \in K[\overline{X}] \mid \exists k \in \mathbb{N} \\ \exists g_i, h_j & \in & K[\overline{X}]\ \ (\alpha f)^{4k} + \sum g_i^4 - \alpha^2 \sum h_j^4 \in \mathcal{P}\} \end{array}$$

where we denote as usual as usual :
$V_K(\mathcal{P}) = \{\overline{x} \in K^n \mid \forall f \in \mathcal{P}\ \ f(\overline{x}) = 0\}$, where $\mathcal{P}$ an ideal of $K[\overline{\mathsf{X}}]$)
$I_K(W) = \{f \in K[\overline{X}] \mid \forall \overline{x} \in W\ \ f(\overline{x}) = 0\}$, where $W \subset K^n$

Note that a Positivstellesatz can be found in [F].

8 Application of $\mathbb{R}-$places in Real geometry

8.1 A criterium for separation of connected components in $M(K)$

Theorem 61 *(Becker-Gondard [BG2]) : Two $\mathbb{R}-$places λ_P and λ_Q are in two distinct components of $M(K)$ if and only if :*

$$\exists b \in K^*\ (b \in P \cap -Q\ \ et\ \beta^2 \in \sum K^4).$$

This criterium is obtained using orderings of higher level more precisely of exact level 2.

$\chi(K) = H(a) \cup H(-a)$ and $H(a) \cap H(-a) = \varnothing$, but $\Lambda(H(a)) \cap \Lambda(H(-a))$ might be non empty.

Nevetherless, if there exist $b \notin \sum K^2$ with $b^2 \in \sum K^4$, then there does not exist $P \in H(b)$ and $Q \in H(-b)$ such that $\lambda_P = \lambda_Q$.

Otherwise $b \notin (P \cap Q) \cup -(P \cap Q)$ and $\lambda_P = \lambda_Q$ imply, as said before that there exist a level 2 ordering P_2, such that

$$P_2 \cup -P_2 = (P \cap Q) \cup -(P \cap Q)$$

with $b \notin P_2 \cup -P_2$, hence $b^2 \notin P_2$, so $b^2 \notin \sum K^4 = \cap P_{2,i}$, where $P_{2,i}$ run over the set of all orderings whose level divides 2.

Assume that λ_P et λ_Q are in the same connected component C of $M(K)$ (with $P \neq Q$), and that there exist $b \in P \cap -Q$ with $b^2 \in \sum K^4$; Λ being closed $C \cap \Lambda(H(b))$, and $C \cap \Lambda(H(-b))$ form a partition of C in two non empty closed sets, impossible.

Conversely :

If λ_P et λ_Q are in C et C', two distinct connected components of $M(K)$, $M(K)$ being a compact Hausdorff space there exist an open-closed $U \supset C$ and $U^c = M(K) \backslash U \supset C'$.

Let $X = \Lambda^{-1}(U)$ and $Y = \Lambda^{-1}(U^c)$; X and Y form a partition of $\chi(K)$; Λ being surjective we get :

$$\Lambda^{-1}(\Lambda(\Lambda^{-1}(U))) = \Lambda^{-1}(U)$$

so $\Lambda^{-1}(\Lambda(X)) = X$, and also $\Lambda^{-1}(\Lambda(Y)) = Y$.

The following lemma from Harman ensure then the existence of b such that $X = H(b)$ and $Y = H(-b)$ with $b^2 \in \sum K^4$, hence we have $b \in P \cap -Q$ with $b^2 \in \sum K^4$.

Harman's Lemma ([H]) : If $\chi(K) = \chi_1 \cup \chi_2$, where χ_1 and χ_2 are disjoint open-closed sets such that $\Lambda^{-1}(\Lambda(\chi_1)) = \chi_1$ *et* $\Lambda^{-1}(\Lambda(\chi_2)) = \chi_2$, then there exist a such that $\chi_1 = H(a)$ and $\chi_2 = H(-a)$.

8.2 Number of connected components of a smooth real variety

Theorem 62 *(Becker-Gondard [BG2]) : Let Y be a smooth non empty projective variety on $\mathbb{R}$, with function field $K = \mathbb{R}(Y)$. Then $\mid \pi_0(Y(\mathbb{R})) \mid$, the number of connected components of $Y(\mathbb{R})$, is given by:*

$$\mid \pi_0(Y(\mathbb{R})) \mid = 1 + \log_2[(K^{*2} \cap \sum K^4) : (\sum K^{*2})^2]$$

This result in the spirit of that of Harnack giving as upper bound of the number of connected components of a smooth projective curve $V(\mathbb{R})$, $g+1$, where g is the genius of V ; but here we give a formula with equality and for any dimension. And the theorem shows clearly the known fact that the number of connected components is a birational invariant among the smooth varieties.

The first proof (1992) of this result is given in [BG2].

Two new proofs have been found in 2003-2004 by Jean-Louis Colliot-Thélène [CT] and by Claus Scheiderer [Sche].

In the original proof the theorem is derived from the two following lemmas which make use of the connected components of the spare of $\mathbb{R}-places\ M(K)$.

Lemma 63 *Let Y be a smooth projective variety on $\mathbb{R}$, with function field $K = \mathbb{R}(Y)$. Then $\mid \pi_0(Y(\mathbb{R})) \mid$ the number of connected components of $Y(\mathbb{R})$ is equal to :*

$$\mid \pi_0(Y(\mathbb{R})) \mid = \mid \pi_0(M(\mathbb{R}(Y))) \mid .$$

Lemma 64 *For any real field K :*

$$\mid \pi_0(M(K)) \mid = 1 + \log_2[(K^{*2} \cap \sum K^4) : (\sum K^{*2})^2].$$

Sketch of proof of first lemma

We use the center map $c : M(K) \to Y(\ \mathbb{R})$, defined by $x = c(\lambda) = c(V_\lambda)$ the unique point (Y projective) whose local ring $\mathfrak{o}_x$ is dominated by V_λ, the valuation ring associated to the $\mathbb{R}-$place λ.

- In this case it is known [e. g. [BCR] Prop. 7.6.2 (ii), p. 133] that c is surjective , the central points being the closure of regular points. And one can prove that c is continuous.

- Bröcker proved in an unpublished manuscript that the fiber of a central point has a finite number of connected components, and that if x is a regular point then the fiber is connected.

Now we just have to use the following topological lemma : if a mapping between two compact spaces X and Y is continuous and surjective and if each fiber is connected then it induces a bijection between $\pi_0(X)$ and $\pi_0(Y)$.

Sketch of proof of second lemma

We prove that $\mid \pi_0(M(K)) \mid = \log_2[E : E^+]$ where E is the group of units of the real holomorphy ring $H(K)$ (defined part 9) and $E^+ = E \cap \sum K^2$.

Then we can prove that the quotient group $(K^{*2} \cap \sum K^4)/(\sum K^{*2})^2$ is isomorphic to $E/(E^+ \cup -E^+)$.

9 $\mathbb{R}-$places and Real Holomorphy Ring

Definition 65 *The real holomorphy ring denoted $H(K)$, is the ring intersection of all real valuation rings of K.*

We also can write $H(K) = \bigcap_{P\in\chi(K)} A(P)$, and

$$H(K) = A(\sum K^2) = \{a \in K \mid \exists n \in \mathbb{N}, n \geq 1 \text{ such that } n \pm a \in \sum K^2\}.$$

$H(K)$ is a Prüfer ring (ring $R \subset K$ such that for any prime ideal p the localized ring R_p is a valuation ring of K), with quotient field K.

In the sequence we shall denote

$$Sper(H(K)) = \{\alpha = (p,\overline{\alpha}),\ p \in specH(K),\ \overline{\alpha} \text{ ordre de } quot(H(K)/p)\}$$

the real spectrum of the real holomorphy ring of K.

Theorem 66 *(Becker-Gondard [6]) :*

The following diagram is commutative :

$$\begin{array}{lcr} \chi(K) \quad \overset{speri}{\longrightarrow} & & MinSperH(K) \\ \downarrow \Lambda & & \downarrow sp \\ M(K) \overset{res}{\to} Hom(H(K),\mathbb{R}) & \overset{j}{\to} & MaxSperH(K) \end{array}$$

where the horizontal mappings are homeomorphisms, and the vertical ones continuous surjective mappings.

The mappings of the above diagram are defined as follow :

$\Lambda : \chi(K) \longrightarrow M(K)$ is given by $P \mapsto \lambda_P$;

$speri : \chi(K) \longrightarrow MinSperH(K)$ is given by $P \mapsto P \cap H(K)$;

$sp : MinSperH(K) \longrightarrow MaxSperH(K)$ is given by $\alpha \longmapsto \alpha^{\max}$ (where $\alpha^{\max}$ is the unique maximal specialization of α) ;

$res : M(K) \longrightarrow Hom(H(K),\mathbb{R})$ is given by $\lambda \mapsto \lambda_{|H(K)}$;

$j : Hom(H(K),\mathbb{R}) \longrightarrow MaxSperH(K)$ is given by $\varphi \mapsto \alpha_\varphi$ (where following the notation for real spectrum, $\alpha_\varphi = \varphi^{-1}(\mathbb{R}^2)$, where $\alpha_\varphi = (\ker\varphi, \overline{\alpha})$ with $\overline{\alpha} = \mathbb{R}^2 \cap quot(\varphi(H(K))$.

All the spaces in the diagram are compact and the topologies of $M(K)$ and $MaxSperH(K)$ are the quotient topologies inherited from Λ and sp.

Hence the space $\chi(K)$ of orderings of a field is homeomorphic to $MinSperH(K)$, and the space $M(K)$ of $\mathbb{R}-$*places* is homeomorphic to $MaxSperH(K)$.

REFERENCES

[AR] C. Andradas et J. Ruiz : *Algebraic and analytic geometry of fans,* Memoirs of AMS, vol.115 #553, (1995).

[BCR]

[Be1]E. Becker **:** *Hereditarily Pythagorean Fields and Orderings of Higher Level,* IMPA Lecture Notes 29, Rio de janeiro (1978).

[Be2] E. Becker : *Valuations and real places in the theory of formally real fields* **et** *The real holomorphy ring and sums of 2^n-th powers*, in LNM 959, Géométrie Réelle et Formes Quadratiques, 1-40 **et** 139-181 (1982).

[Be3] E. Becker : *Extended Artin-Schreier theory of fields,* Rocky Mountain J. of Math., vol 14 #4, 881-897 (1984).

[BBG] E. Becker, R. Berr, et D. Gondard : *Valuation fans and residually real-closed henselian fields,* J. of Algebra, 215, 574-602 (1999).

[BBDG] E. Becker, R. Berr, F. Delon, et D. Gondard : *Hilbert's 17th problem for sums of 2nth powers,* J. Reine Angew. Math. 450, 139-157 (1994).

[BG1] E. Becker and D. Gondard : *On rings admitting orderings and 2-primary chains of orderings of higher level,* Manuscripta Math. 65, 63-65 (1989).

[BG2] E. Becker et D. Gondard : *Notes on the space of real places of a formally real field,* in RAAG (Trento), W. de Gruyter, 21-46 (1995).

[Br] R. Brown : *Automorphism and isomorphism of henselian fields,* Trans. A.M.S. 307, 675-703 (1988).

[BCP] R. Brown, T. Craven, M. Pelling : *Ordered fields satifying Rolle theorem,* Ill J. Math. 30, 66-78 (1980).

[CT] J.–L. Colliot Thélène : *Eine Bemerkung zu einem Satz von E. Becker und D. Gondard*, http://www.math.psud.fr/~colliot/ ; prépublication #5, à paraître dans Math. Zeitschrift.

[De1] F. Delon : *Corps et anneaux de Rolle*, Proc. Amer. Math. Soc. 97, 1986, 315-319 (1986).

[De2] F. Delon : *Compléments sur les corps chaîne-clos,* in séminaire Structures Algébriques Ordonnées, Publications de l'Université paris VII, vol 33, 13-20 (1990).

[DG] F. Delon and D. Gondard : *17ème problème de Hilbert sur les corps chaîne-clos*, J.S.L. 56, #3, 853-861 (1991).

[DF] F. Delon and R. Farre : *Some model theory for almost real-closed fields,* J.S.L. 61 #4, 1121-1151 (1996).

[Di] M. Dickmann : *The model theory of chain-closed fields,* J. S.L. 53, 921-930, (1988).

[F] R. Farré : *A Positivstellensatz for chain-closed fields,* $\mathbb{R}((t))$ *and some related fields,* Arch. Math. vol 57, 446-455 (1991).

[G1] D. Gondard-Cozette : *Axiomatisation simple des corps de Rolle,* Manuscripta Mathematica 69, 267-274 (1990).

[G2] D. Gondard-Cozette : *Chainable fields and real algebraic geometry,* in LNM 1420, M. Galbiati and A. Tognoli eds., 128-148, Springer-verlag, *(1990).*

[G3] D. Gondard : *Théorie du premier ordre des corps chaînables et des corps chaîne-clos,*C. R. Acad. Sc. Paris t. 304 , # 16 (1987).

[G4] D. Gondard-Cozette : $\mathbb{R}-$*places et Géométrie algébrique réelle,* in Séminaire Structures Algébriques Ordonnées, prépublication # 75, Equipe de Logique Université Paris VII, mai 2003, soumis.

[H] J. Harman : *Chains of higher level orderings,* Contemporary Mathematics, vol 8, 141-174, A.M.S.,(1982).

[J1] B. Jacob : *Fans, real valuations, and hereditarily-pythagorean fields,* Pacific J. Math. 93 #1, 95-105 (1981).

[J2] B. Jacob : *The model theory of generalized real-closed fields,* J. reine angew. Math 323, *213-220 (1981).*

[L] T. Y. Lam : *Orderings, Valuations and quadratic forms,* AMS, regional conference series in mathematics #52*, (1983).*

[S1] N. Schwarz : *Chain signatures and real closure*, J. reine angew. Math 347, 1-20, (1984).

[S2] N. Schwartz : *Signatures and real closures of fields,* in Séminaire Structures Algébriques Ordonnées, Publications de l'Université Paris 7, vol 33, 65-78 (1990).

[Sche] C. Scheiderer : *A short remark on a theorem by Becker and Gondard,* http://www.uni-duisburg.de/FB11/FGS/F1/claus.html#notes

[Schü1] H.-W. Schülting : *On real places of a field and their holomorphy ring* , Communications in Algebra, 10, 1239-1284 (1982).

[Schü2] H.-W. Schülting : *The strong topology on real algebraic varieties,* Contemporary Mathematics 8, 141-174, (1982).

[Schü3] H.-W. Schülting : *Real holomorphy rings in real algebraic geometry,* LNM 959, Géométrie Réelle et Formes Quadratiques, 1-40 **et** 139-181, (1982).

Danielle Gondard-Cozette
gondard@math.jussieu.fr
Institut de Mathématiques de Jussieu (UMR 7586 du CNRS)
Université Paris VI, Case 247
4, Place Jussieu
75252 Paris cedex 05 (France)