On the inverse limit of localizations
February 2004

Introduction Let A be a commutative ring. Its spectrum will here be considered as a mere ordered set, the order relation being the *opposite* of the inclusion of prime ideals. For any A-module M, the inclusion $\mathfrak{q} \subset \mathfrak{p}$ gives rise to a localization map $M_\mathfrak{p} \to M_\mathfrak{q}$, and we are interested in the inverse limit of this system. We will prove

Suppose there exists a finite set of prime ideals, $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n$ *such that the canonical map*

$$M \quad \longrightarrow \quad M_{\mathfrak{p}_1} \times \ldots \times M_{\mathfrak{p}_n}$$

is injective. Then the map

$$M \quad \longrightarrow \quad \text{inv.lim}_{\mathfrak{p} \in \text{Spec}(A)} M_\mathfrak{p}$$

is an isomorphism.

This result was announced without proof in

D. FERRAND, *Sur les modules qui sont limite projective de leurs localisés* C. R. Acad. Sc. Paris, t.262, p.609-611 (14 mars 1966).

During a stay at the Royal Institute of Technology (K.T.H.) in Stockholm, I had a long discussion with Fredrick NORDSTRÖM, who obtains similar results by other methods (see his Master's Thesis *Recovering a module from its local structure*, Dept. Math., KTH, Stockholm, 2003.)
That decided me on writing out proofs. In fact my 1966' proofs rested on methods quite similar to Nordström's (namely, given $(x(\mathfrak{p})) \in \text{inv.lim} M_\mathfrak{p}$, each $x(\mathfrak{p}) \in M_\mathfrak{p}$ can be extended as a section of the sheaf $\tilde{M}$ over an open set U, containing $\mathfrak{p}$, and U can be taken small enough for this section to be sent to $x(\mathfrak{q}) \in M_\mathfrak{q}$ for all $\mathfrak{q} \in U$. Then glue these sections). Later on, it appeared to me that the use of faithfully flat descent brought simplifications, and finally I propose here a completely elementary proof.

Definitions and notations are those from

N. BOURBAKI, *Commutative Algebra, Chapters 1-7*, Springer, 1989 (cited AC)

Example The classical case of a torsionless module M over an integral domain A deserves to be mentioned. Let K be the fraction field of A. The torsionless hypothesis means that the map $M \longrightarrow K \otimes_A M = M_{(0)}$ is injective; it implies that all the the maps $M_\mathfrak{p} \longrightarrow K \otimes_A M$ are also injective. Thus we may identify the modules M and $M_\mathfrak{p}$ with their image in $K \otimes_A M$; then the inverse limit may be seen as the mere intersection $\bigcap M_\mathfrak{p}$. In that case, the equality

$$M = \bigcap M_\mathfrak{p}$$

is well known (see AC II 3.3 Cor. 4, or the discussion in GODEMENT, *Théorie des faisceaux*, p.124).

In the case where the ring A is noetherian, the hypothesis on the module, in the statement of the introduction, is fulfilled when M is of finite type because we can then take for the $\mathfrak{p}_i$ the prime ideals associated with M (AC IV 1.4 Cor.).

For an A-module M, and a subset $Z \subset \mathrm{Spec}(A)$ endowed with the induced ordering, we let

$$M(Z) := \mathrm{inv.lim}_{\mathfrak{p} \in Z} M_{\mathfrak{p}}.$$

It may be sometimes usefull to look at $M(Z)$ as a submodule of the product $\prod_{\mathfrak{p} \in Z} M_{\mathfrak{p}}$. If $Z' \subset Z$, there is clearly a "projection map" $M(Z) \to M(Z')$.

Lemma 1 *If A is a local ring, or a finite product of local rings, then for any A-module M, the canonical map*

$$M \quad \longrightarrow \quad M(\mathrm{Spec}(A))$$

is an isomorphism.□

Lemma 2 *For any module M, the canonical map $M \longrightarrow M(\mathrm{Spec}(A))$ is injective.*

An element $x \in M$ is in the kernel of this map if for every prime ideal $\mathfrak{p}$ the image of x in $M_{\mathfrak{p}}$ is zero, that is if there exists $s \in A - \mathfrak{p}$ such that $sx = 0$. Therefore the ideal $\mathrm{Ann}(x)$ is not contained in any prime ideal. Therefore $\mathrm{Ann}(x) = A$, and thus $x = 0$.(For another proof, see AC II 3.3 Cor.2) □

Proposition 1 *Let $f : A \to B$ be a morphism of commutative rings. Let N be a B-module and let denote $f_*(N)$ the underlying A-module. Then the canonical map*

$$\varphi : f_*(N)(\mathrm{Spec}(A)) \longrightarrow N(\mathrm{Spec}(B))$$

is injective. In particular, if $N \to N(\mathrm{Spec}(B))$ is an isomorphism, then $f_*(N) \to f_*(N)(\mathrm{Spec}(A))$ is also an isomorphism.

Proof. No confusion can arise in denoting simply by N the A-module $f_*(N)$. The canonical map φ is defined as follows: for a prime ideal $\mathfrak{q} \in \mathrm{Spec}(B)$, $f^{-1}(\mathfrak{q})$ is a prime ideal of A, and there is a canonical A-linear map $N_{f^{-1}(\mathfrak{q})} \to N_{\mathfrak{q}}$; by composing it with the projection

$$N(\mathrm{Spec}(A)) = \mathrm{inv.lim}_{\mathfrak{p} \in \mathrm{Spec}(A)} N_{\mathfrak{p}} \longrightarrow N_{f^{-1}(\mathfrak{q})}$$

we get an A-linear map $N(\mathrm{Spec}(A)) \longrightarrow N_{\mathfrak{q}}$ which, for inclusions $\mathfrak{q}' \subset \mathfrak{q}$, clearly gives rise to the expected commutative triangles. By definition of the inverse limit we thus get the map

$$\varphi : N(\mathrm{Spec}(A)) \longrightarrow N(\mathrm{Spec}(B)).$$

In order to prove the injectivity of φ, let us consider $\xi \in N(\mathrm{Spec}(A))$ such that $\varphi(\xi) = 0$. One has to check that, for every prime ideal $\mathfrak{p} \in \mathrm{Spec}(A)$, the projection of ξ in $N_{\mathfrak{p}}$ is zero. But $N_{\mathfrak{p}} \to N(\mathrm{Spec}(A_{\mathfrak{p}}))$ is an isomorphism since $A_{\mathfrak{p}}$ is local. Using now the set $\mathrm{Spec}(B_{\mathfrak{p}}) = \{\mathfrak{q} \in \mathrm{Spec}(B), f^{-1}(\mathfrak{q}) \subset \mathfrak{p}\}$, we get the commutative square

$$\begin{array}{ccc} N(\mathrm{Spec}(A)) & \xrightarrow{\varphi} & N(\mathrm{Spec}(B)) \\ \downarrow & & \downarrow \\ N_{\mathfrak{p}} = N(\mathrm{Spec}(A_{\mathfrak{p}})) & \longrightarrow & N(\mathrm{Spec}(B_{\mathfrak{p}})) = N_{\mathfrak{p}}(\mathrm{Spec}(B_{\mathfrak{p}})). \end{array}$$

But the lower arrow is injective by Lemma 2 applied to the $B_{\mathfrak{p}}$-module $N_{\mathfrak{p}}$; whence the result.□

Lemma 3 *Let $M \subset N$ be an inclusion of A-modules. If the canonical map $N \to N(\mathrm{Spec}(A))$ is an isomorphism, then the map $M \to M(\mathrm{Spec}(A))$ is also an isomorphism.*

By assumption, for $(x(\mathfrak{p})) \in M(\mathrm{Spec}(A))$, there exists a unique $y \in N$ whose image in $N_{\mathfrak{p}}$ is equal to $x(\mathfrak{p})$, and we have to show that $y \in M$. Let I denote the ideal of those elements $a \in A$ such that $ay \in M$. The conclusion is equivalent to $I = A$. But for each prime ideal $\mathfrak{p}$, $I_{\mathfrak{p}}$ is the ideal of the $a' \in A_{\mathfrak{p}}$ such that $a'y \in M_{\mathfrak{p}}$, thus, by assumption, $I_{\mathfrak{p}} = A_{\mathfrak{p}}$ (see AC II 3.3 Cor.1 p.88).□

Proposition 2 *Let $f : A \to B$ be a morphism of commutative rings, and let M be an A-module. Suppose that the following conditions hold :*
a) The canonical map $M \to B \otimes_A M$ is injective.
b) The map $B \otimes_A M \to (B \otimes_A M)(\mathrm{Spec}(B))$ is an isomorphism.
Then $M \to M(\mathrm{Spec}(A))$ is an isomorphism.

By condition a) and Lemma 3, it is enough to check that the map $B \otimes_A M \to (B \otimes_A M)(\mathrm{Spec}(A))$ is an isomorphism, but that comes from condition b) and proposition 1.□

Corollary *If A is a semi-local ring, then for any module M, the map*

$$M \to M(\mathrm{Spec}(A))$$

is an isomorphism.

Proof. Let $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n$ be the maximal ideals of A, and let $B = \prod_1^n A_{\mathfrak{p}_i}$. The morphism $A \to B$ is faithfully flat (AC I 3.5), therefore condition a) is fulfilled. Condition b) trivially holds since B is a finite product of local rings.□

Proposition 3 *Let M be an A-module. Suppose there exist a finite set of prime ideals, $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n$ such that the canonical map*

$$M \quad \longrightarrow \quad M_{\mathfrak{p}_1} \times \ldots \times M_{\mathfrak{p}_n}$$

is injective. Then the map

$$M \quad \longrightarrow \quad \mathrm{inv.lim}_{\mathfrak{p} \in \mathrm{Spec}(A)} M_{\mathfrak{p}}$$

is an isomorphism.

Proof. Let $B = \prod_1^n A_{\mathfrak{p}_i}$; condition a) of the previous proposition holds by assumption, and condition b) is again trivially true.