A note on transfer

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1. Introduction

Let G be a subgroup of finite index of a group E. A century ago, I. Schur invented the *tranfer homomorphism*

$$\mathrm{Ver} : E^{\mathrm{ab}} \longrightarrow G^{\mathrm{ab}},$$

where G^{ab} denotes the maximal abelian quotient $G/D(G)$ of G, the subgroup $D(G)$ being generated by the commutators of elements of G. The transfer is, since then, of current use in group theory, and elsewhere (see any textbook on finite groups, for example [S 2], ch. 7).

Among the many interpretations of this homomorphism, the one due to Cartier ([Ca]) indicates a path for what follows; he actually noticed that the tranfer rests on only two data :
- The action of G on E by right multiplication, making E a free (right) G-set.
- The left action of E on itself by left multiplication, allowing to see E as a group of automorphisms of the right G-set E.

Here is the definition of the transfer using these data alone.
Let X be a set acted on freely (on the right) by a group G, and let $\mathrm{Aut}_G(X)$ be the group of automorphisms of this G-set. We suppose that the set X/G is finite. We can then define a group homomorphism

$$\mathrm{Ver} : \mathrm{Aut}_G(X) \longrightarrow G^{\mathrm{ab}}$$

as follows. Let T be a "representative subset" for the quotient $\pi : X \to X/G$, that is a subset $T \subset X$ on which π induces a bijection : $\pi : T \xrightarrow{\sim} X/G$; any $x \in X$ may thus be written uniquely as $x = tg$ with $t \in T$ and $g \in G$. Let now $u : X \to X$ be a G-automorphism; it induces a bijection $\bar{u} : X/G \to X/G$, which can be lifted as a bijection σ of T : $\pi(\sigma t) = \bar{u}(\pi t)$, and thus $\pi(ut) = \pi(\sigma t)$. Therefore, there exists a unique $g_{\sigma(t)} \in G$ such that

$$u(t) \;=\; \sigma(t) g_{\sigma(t)}.$$

Denoting by g^{ab} the image of an element g in the quotient G^{ab}, we let

$$\mathrm{Ver}_T(u) \;=\; \prod_{t\in T} g_t^{\mathrm{ab}}$$

It can be checked (see [S 2], p.51) that this product does not depend of the choice of the representative subset T, and that it defines a homomorphism $\mathrm{Aut}_G(X) \longrightarrow G^{\mathrm{ab}}$. It is the (extended) transfer. In the initial situation of a subgroup G of finite index of a group E, seen as a right G-set, the left multiplication in E gives a homomorphism $E \to \mathrm{Aut}_G(E)$, and the composition $E \to \mathrm{Aut}_G(E) \to G^{\mathrm{ab}}$ is the usual transfer.

The first result of this note is the remark that the usual transfer homomorphism may be extended as an *isomorphism*; for stating it we also need the *signature*, that is the homomorphism

$$\mathrm{Aut}_G(X) \longrightarrow \mathrm{Aut}(X/G) \xrightarrow{\mathrm{sgn}} \{\pm 1\}.$$

(Recall that for a finite set Q of cardinal ≥ 2, the signature gives an isomorphism $\mathfrak{S}(Q)^{\mathrm{ab}} \simeq \{\pm 1\}$).

Theorem (Set theoretical form) *Let G be a group acting freely on a set X. Suppose that the quotient X/G is a finite set not reduced to a point. Then, the transfer and the signature induce an* isomorphism

$$\mathrm{Aut}_G(X)^{\mathrm{ab}} \xrightarrow{(\mathrm{Ver},\,\mathrm{sgn})} G^{\mathrm{ab}} \times \{\pm 1\}.$$

The main aim of this note is to extend this combinatorial statement, and its proof, to a topos setting (in the Grothendieck sense).

Theorem (6.1) *Let E be a topos. Let G be a group in E, and let X be an object in E endowed with a free (right) action of G. We suppose that the quotient X/G is locally constant finite, and is not the final element of E. Then there is an isomorphism of groups in E*

$$\mathsf{Aut}_G(X)^{\mathrm{ab}} \quad \xrightarrow{\sim} \quad G^{\mathrm{ab}} \times \{\pm 1\}.$$

Here, G^{ab} denotes the (object of E representing the) sheaf associated with the presheaf $U \mapsto \mathrm{Hom}_{\mathsf{E}}(U,G)^{\mathrm{ab}}$.

In a first step we consider the (excluded) case of a (right) G-torsor X in E, that is the case where X/G is the final element of E ; the group $\mathsf{Aut}_G(X)$ is then the classical *adjoint group*, or the group G *twisted by the torsor* X, and is also denoted by $X \wedge^G G$; in general the group G and its adjoint group are not isomorphic ; but, since the action of G on the second factor of $X \wedge^G G$ is given by inner automorphisms, it is pretty clear that the groups $(X \wedge^G G)^{\mathrm{ab}}$ and G^{ab} are indeed isomorphic. The §3 is devoted to a careful study of this isomorphism ; but since a right G-torsor X is a $(\mathsf{Aut}_G(X), G)$-bitorsor, it is more natural to consider from scratch a pair of groups and a (H,G)-bitorsor X, and to explain how it gives rise to a group isomorphism

$$\alpha_X : H^{\mathrm{ab}} \xrightarrow{\sim} G^{\mathrm{ab}}.$$

As a consequence we get that if two objects in a topos are locally isomorphic, then the abelianizations of their automorphism groups are isomorphic (**3.3**). In §4, we extend that point to define easily the abelianization of a transitive groupoïd in the topos E.

Under the hypotheses of the theorem, when X/G is no longer the final element, but is locally constant finite, then the G-objects X and $(X/G) \times G$ are locally isomorphic, and the result (**3.3**) quoted above allows to "calculate" $(\mathsf{Aut}_G(X))^{\mathrm{ab}}$, thus getting a proof of the theorem.

In §7, we check that, when G is abelian, the transfer is induced by the determinant of the $\mathbf{Z}G$-linear map $\mathbf{Z}u : \mathbf{Z}X \longrightarrow \mathbf{Z}X$:

$$\det(\mathbf{Z}u) \;=\; \mathrm{Ver}(u).\mathrm{sgn}(\bar{u}).$$

A last paragraph is slightly on the fringes of what precedes. It deals with the extensions of groups, but it uses the circle of methods introduced in the previous parts.

The letter E will denote a topos in the Grothendieck sense, with final element e_{E} or e. I will stick to the definitions and notations from Grothendieck-Verdier [Gr-V] *and* Giraud [Gi] *; in particular, the symbol* Hom *denotes a* set *of arrows, while* Hom *denotes an object in E ; the object $\mathsf{Hom}(X,Y)$ represents the functor $Z \mapsto \mathrm{Hom}(Z \times X, Y)$. Some notions on toposes are recollected in the appendix.*

2. Torsors and bi-torsors

2.1. Definition *Given a group G in E, a (right) G-torsor is an object X endowed with a right action of G*

$$m : X \times G \quad \longrightarrow \quad X,$$

such that
i) $X \longrightarrow e$ *is a covering;*
ii) the map

$$X \times G \quad \longrightarrow \quad X \times X, \qquad (x,g) \longmapsto (x, m(x,g))$$

is an isomorphism.
Given another group H *in* E*, a* (H,G)*-bitorsor is a right* G*-torsor* X *endowed with a left action of* H*, which commutes with the action of* G*, and making* X *a (left)* H*-torsor.*([Gi], III, 1.4.1, and 1.5.1)

The condition *i)* is equivalent to the useful variant

i') *There exists a covering* $S \longrightarrow e$ *such that* $S \times X \longrightarrow S$ *admits a section.*

2.2 Lemma *Let* X *be a right* G*-object in the topos* E*.*
i) If X *is a torsor, then the morphism* $X/G \longrightarrow e$ *is an isomorphism.*
ii) Conversely, if the action of G *is free,* i.e *if* $X \times G \to X \times X$ *is a monomorphism, and if* $X/G \to e$ *is an isomorphism, then* X *is a* G*-torsor.*

i) Since the map $X \to e$ is a covering, by (A.3.1) the sequence $X \times X \underset{pr_2}{\overset{pr_1}{\rightrightarrows}} X \longrightarrow e$ is exact. But the action of G on X induces an isomorphism $X \times G \longrightarrow X \times X$, and by definition of the quotient, the sequence $X \times G \rightrightarrows X \longrightarrow X/G$ is also exact (A.4). So $X/G \longrightarrow e$ is an isomorphism.

ii) Free actions are discussed below (5.2), where the map $X \times G \longrightarrow X \times_{X/G} X$ is shown to be an isomorphism; if moreover $X/G \simeq e$, then X is indeed a G-torsor.

2.3 Adjoint group

Let G be a group in E, and let X be a right G-torsor. Consider the group $\mathsf{Aut}_G(X)$ of G-automorphisms of X i.e the sheaf of automorphisms $\alpha : X \to X$ such that $\alpha(xg) = \alpha(x)g$; it is sometimes called the adjoint group ([Gi] III 1.4.8, here denoted by $\mathsf{ad}(X)$) or the group "twisted by" X (it is then denoted by ${}^X G$) ([Gi] III 2.3.7), or the *gauge group*. Recall that the isomorphism ${}^X G \xrightarrow{\sim} \mathsf{Aut}_G(X)$ is defined element-wise as follows : given $(x,g) \in X \times G$, there is a unique G-automorphism u of X such that $u(x) = xg$, and thus $u(xh) = xgh = (xh)(h^{-1}gh)$; therefore (x,g) and $(xh, h^{-1}gh)$ define the same automorphism; hence a map $X \wedge^G G \longrightarrow \mathsf{Aut}_G(X)$, where the exponent G operates on the group G by conjugation. This map is easily seen to be an isomorphism.

2.3.1 Lemma *A right* G*-torsor* X *is a left* $\mathsf{Aut}_G(X)$*-torsor. In particular, if* X *is an* (H,G)*-bitorsor, the action of* H *gives a group isomorphism* $\lambda_X : H \simeq \mathsf{Aut}_G(X)$*. Finally, if* G *is abelian, the right action is also a left action, the right torsor* X *is a* (G,G)*-bitorsor, and one has an isomorphism* $\lambda_X : G \simeq \mathsf{Aut}_G(X)$*.*

We have to check that the map $\mathsf{Aut}_G(X) \times X \longrightarrow X \times X$ given by $(u,x) \mapsto (u(x), x)$ is an isomorphism. This may be done after the base change by the covering $X \longrightarrow e$ (**A.3.3**). But then, the right G-torsor X becomes isomorphic to G_r, i.e G with the action given by multiplication on the right; but the multiplication on the left gives a group isomorphism $G \to \mathsf{Aut}_G(G_r)$, and the map $G \times G_r \longrightarrow G_r \times G_r$, $(g,x) \mapsto (gx, x)$ is clearly an isomorphism.
If, now, X is an (H,G)-bitorsor, the map $\lambda_X : H \to \mathsf{Aut}_G(X)$ fits into the commutative diagram

$$\begin{tikzcd} H \times X \arrow[r, "\lambda_X \times 1"] \arrow[d] & \mathsf{Aut}_G(X) \times X \arrow[d] \\ X \times X \arrow[r, equal] & X \times X \end{tikzcd}$$

where the vertical maps are isomorphisms; the conclusion follows again from the fact that $X \to e$ is a covering (**A.3.3**).

2.4 We denote by $\mathsf{Gr}_{\mathrm{iso}}(\mathsf{E})$ the category of groups in E, with group isomorphisms as only arrows. It contains the subcategory $\mathsf{Ab}_{\mathrm{iso}}(\mathsf{E})$ whose objects are abelian groups.

2.5 We denote by $\mathsf{Gr}_{\mathrm{Mrt}}(\mathsf{E})$ the category whose objects are the groups in E, a map from a group H to a group G being an *isomorphism class* of (H,G)-bitorsors. Given a map from G_1 to G_2, and a map from G_2 to G_3, that is a (G_1,G_2)-bitorsor X and a (G_2,G_3)-bitorsor Y, the composite map from G_1 to G_3 is associated to the (G_1,G_3)-bitorsor $X\wedge^{G_2}Y$ (When the group G_2 is clear from the context we omit it, and we simply write $X\wedge Y$) . Thus, if we note by $[X]$ the map associated with the bitorsor X, the formula for composition is

$$[Y]\circ[X] \;=\; [X\wedge Y].$$

(Note the interchange between X and Y). The subscript "Mrt" refers, of course, to Morita.

2.6 A functor $\mathsf{Gr}_{\mathrm{iso}}(\mathsf{E}) \longrightarrow \mathsf{Gr}_{\mathrm{Mrt}}(\mathsf{E})$ is defined by sending an isomorphism $u: H\to G$ to the (H,G)-bitorsor ${}_uG$. Let v be an other isomorphism $H\to G$; if the (H,G)-bitorsors ${}_uG$ and ${}_vG$ are isomorphic, then $v=\mathrm{int}_g\circ u$, where g is the "global section" of G image of the unit element by the given isomorphism.

As shown below, any (H,G) bitorsor is locally isomorphic to a bitorsor of the form ${}_uG$, for locally given isomorphisms u.

3. The functor $\mathsf{Gr}_{\mathrm{Mrt}} \longrightarrow \mathsf{Ab}_{\mathrm{iso}}$.

3.1. For G a (discrete) group, we write G^{ab} for the greatest abelian quotient of G, that is the quotient $G/D(G)$, where $D(G)$, the "derived subgroup" of G, is *generated by* the commutators of elements of G.

Let now G be a group in a topos E; we denote by G^{ab} the object of E representing the sheaf associated to the presheaf $X\longmapsto \mathrm{Hom}_{\mathsf{E}}(X,G)^{\mathrm{ab}}$.
According to (**A.5**), G^{ab} is characterized by the following property : there is a map of functors in X, $\mathrm{Hom}(X,G)^{\mathrm{ab}}\longrightarrow \mathrm{Hom}(X,G^{\mathrm{ab}})$ such that for any $Y\in\mathsf{E}$, and any map of functors in X, $\mathrm{Hom}(X,G)^{\mathrm{ab}}\longrightarrow \mathrm{Hom}(X,Y)$ there is a unique map $G^{\mathrm{ab}}\longrightarrow Y$ making the obvious triangle commutative.

3.2 Theorem *Let E be a topos. There exists a functor*

$$\alpha:\mathsf{Gr}_{\mathrm{Mrt}}(\mathsf{E}) \longrightarrow \mathsf{Ab}_{\mathrm{iso}}(\mathsf{E}),$$

which sends a group G to its abelianization G^{ab}, and which transforms a (H,G)-bitorsor X in a group isomorphism $\alpha_X: H^{\mathrm{ab}}\simeq G^{\mathrm{ab}}$
If $X={}_uG$ is the bitorsor associated with an isomorphism of groups $u:H\to G$, then α_X is the isomorphism $u^{\mathrm{ab}}: H^{\mathrm{ab}}\to G^{\mathrm{ab}}$ induced by u.

Let X be a (H,G)-bitorsor. First let us show how to associate with X a group isomorphism $\alpha_X: H^{\mathrm{ab}}\longrightarrow G^{\mathrm{ab}}$. By assumptions, the operations of H and G give isomorphisms

$$H\times X \;\xrightarrow{\sim}\; X\times X \;\xleftarrow{\sim}\; X\times G.$$

They may be described as $(h,x)\mapsto(hx,x)$ and $(x,g)\mapsto(xg,x)$. By composition, we get an isomorphism $\theta: H\times X\longrightarrow X\times G$ over X, i.e the following triangle is commutative :

(3.2.1)
$$\begin{tikzcd} H\times X \arrow[rr, "\theta"] \arrow[dr, "\mathrm{pr}_2"'] & & X\times G \arrow[dl, "\mathrm{pr}_1"] \\ & X & \end{tikzcd}$$

This isomorphism is characterized as follows : let T be an object of E; for $(h,x)\in H(T)\times X(T)$, the element $g\in G(T)$ such that $\theta(h,x)=(x,g)$ is uniquely defined by the relation

(3.2.2)
$$hx \;=\; xg.$$

The map θ is in fact an isomorphism of *groups over* X since the relations $hx=xg$ and $h'x=xg'$ implie $hh'x=hxg'=xgg'$.

From θ, we get an isomorphism of abelian groups over X

$$\begin{tikzcd} H^{\mathrm{ab}} \times X \arrow[rr, "\theta^{\mathrm{ab}}"] \arrow[dr] & & X \times G^{\mathrm{ab}} \arrow[dl] \\ & X & \end{tikzcd}$$

It remains to descend this isomorphism from X to e_{E}. Since $X \to e_{\mathsf{E}}$ is a covering it is enough to check that the two inverse images of θ^{ab}, relative to the projections

$$X \times X \overset{pr_1}{\underset{pr_2}{\rightrightarrows}} X$$

are equal (**A.3.2**). But the two inverse images of θ itself

$$H \times X \times X \overset{\theta_1}{\underset{\theta_2}{\rightrightarrows}} X \times X \times G$$

may be written as

$$(h, x, y) \longmapsto (x, y, g_1), \quad (h, x, y) \longmapsto (x, y, g_2)$$

with the relations $hx = xg_1$ and $hy = yg_2$; since $X(T)$ is a right $G(T)$-torsor, there exists a unique $g \in G(T)$ such that $y = xg$; from this we see that $g_2 = g^{-1}g_1g$; so g_1 and g_2 have the same image in $G^{\mathrm{ab}}(T)$; hence $\theta_1^{\mathrm{ab}} = \theta_2^{\mathrm{ab}}$, and we are done.

Let X be a (H, G)-bitorsor ; we have to show that α_X depends only of the isomorphism class of X. But let $u : X \to X'$ be an isomorphism of (H, G)-bitorsors. The equality $\alpha_X = \alpha_{X'}$ we are looking for clearly proceeds from the commutativity of the two following squares

$$\begin{tikzcd} H \times X \arrow[r] \arrow[d, "1_H \times u"'] & X \times X \arrow[d, "u \times u"'] & X \times G \arrow[l] \arrow[d, "u \times 1_G"] \\ H \times X' \arrow[r] & X' \times X' & X' \times G \arrow[l] \end{tikzcd} .$$

Now, in order to show that α is a functor, we must check the equality $\alpha_{X \wedge Y} = \alpha_Y \circ \alpha_X$; if X is a (G_1, G_2)-bitorsor, we denote by

$$\theta_X : G_1 \times X \longrightarrow X \times G_2$$

the isomorphism (3.2.1) considered in the beginning of the proof, and we use the analogous notation for Y and $X \wedge Y$. The point to be checked is that the following diagram is commutative.

$$\begin{tikzcd} G_1 \times X \times Y \arrow[r, "\theta_X \times 1_Y"] \arrow[d] & X \times G_2 \times Y \arrow[r, "1_X \times \theta_Y"] & X \times Y \times G_3 \arrow[d] \\ G_1 \times X \wedge Y \arrow[rr, "\theta_{X \wedge Y}"'] & & X \wedge Y \times G_3 \end{tikzcd}$$

Starting with (g_1, x, y) in the upper left corner $G_1(T) \times X(T) \times Y(T)$, its horizontal images may be written as

$$(g_1, x, y) \longmapsto (x, g_2, y) \longmapsto (x, y, g_3)$$

with the relations

$$g_1x = xg_2, \qquad g_2y = yg_3$$

On the lower line, the map $\theta_{X \wedge Y}$ sends $(g_1, x \wedge y)$ to $(x \wedge y, g_3')$, with the relation

$$g_1(x \wedge y) = (x \wedge y)g_3'.$$

But the very definition of the (G_1, G_3)-bitorsor $X \wedge^{G_2} Y$ gives the equalities

$$g_1(x \wedge y) = (g_1x) \wedge y = (xg_2) \wedge y = x \wedge (g_2y) = x \wedge (yg_3) = (x \wedge y)g_3$$

showing that $g_3' = g_3$.

Finally consider an isomorphism $u : H \to G$, and the (H, G)-bitorsor $X = {}_uG$, and let's check that α_X is induced by u. For maps $x, g \in G(T)$ and $h \in H(T)$, the relation defining θ (3.2.2)

$$u(h)x = xg$$

may also be written in the group $G(T)$ as

$$g = x^{-1}u(h)x,$$

showing that g and $u(h)$ have the same image in G^{ab} ; hence the assertion.□

3.3 Corollary *Let X_1 and X_2 be two objects in a topos E. We suppose that they are locally isomorphic, i.e. that there exists a covering $Z \to e_{\mathsf{E}}$ and an isomorphism over Z, $Z \times X_1 \xrightarrow{\sim} Z \times X_2$. Then one has an isomorphism in E*

$$\mathsf{Aut}(X_2)^{\mathrm{ab}} \quad \xrightarrow{\sim} \quad \mathsf{Aut}(X_1)^{\mathrm{ab}}$$

By assumption, the map $Z \to e$ factors through $\mathsf{Isom}_{\mathsf{E}}(X_1, X_2)$; since $Z \to e$ is a covering, $\mathsf{Isom}_{\mathsf{E}}(X_1, X_2) \longrightarrow e$ is also a covering. Moreover the sheaf $\mathsf{Isom}_{\mathsf{E}}(X_1, X_2)$ is a $(\mathsf{Aut}(X_2), \mathsf{Aut}(X_1))$-bitorsor, and we can apply the theorem.

The paragraph 4 contains an amplification of this result to groupoïds in E.

3.4 Example Let Γ be a (discrete) group and $\mathsf{E} = B\Gamma$ be the topos of left Γ-sets, denoted by B_Γ in [Gr-V] p.314, and [Gi], VIII 4.1.

3.4.1 We will define a right G-torsor X in E, where G is a constant group, such that $\mathsf{Aut}_G(X)$ is not constant.
Let first recall how to describe the sheaves $\mathsf{Hom}_{\mathsf{E}}$ in this topos. Let X and Y be two Γ-sets. Then $\mathsf{Hom}_{\mathsf{E}}(X, Y)$ is the set $\mathrm{Hom}_{\mathbf{Sets}}(X, Y)$ of all set-theoretical maps $u : X \to Y$ endowed with the "usual" operation of $\sigma \in \Gamma$, that is ${}^{\sigma}u = \sigma_Y \circ u \circ \sigma_X^{-1}$; if fact, by definition of $\mathsf{Hom}_{\mathsf{E}}$ ([Gr-V], p.491), we must have, for any Γ-set Z, a bijection

$$\mathrm{Hom}_{B\Gamma}(Z \times X, Y) \xrightarrow{\sim} \mathrm{Hom}_{B\Gamma}(Z, \mathsf{Hom}(X, Y))$$

But, starting with a map $f : Z \times X \to Y$ such that $f(\sigma z, \sigma x) = \sigma f(z, x)$, consider the associated map $Z \to \mathrm{Hom}_{\mathbf{Sets}}(X, Y)$, $z \longmapsto u = (x \mapsto f(z, x))$; it sends σz to $(x \mapsto f(\sigma z, x) = \sigma f(z, \sigma^{-1}x) = ({}^{\sigma}u)(x))$; hence the claim.

Let $\rho : \Gamma \longrightarrow G$ be a homomorphism of groups. Let $X = \rho^{\star}(G)$ be the left Γ-set G with operation of $\sigma \in \Gamma$ through ρ : ${}^{\sigma}x = \rho(\sigma)x$. Still denote by G the constant group in $B\Gamma$ defined by G (trivial action of Γ). The product makes X a right G-torsor in $B\Gamma$, since the map $X \times G \longrightarrow X \times X$ is in fact the map $G \times G \longrightarrow G \times G$, $(x, g) \mapsto (x, xg)$, which is clearly bijective and Γ-equivariant.
The sheaf $\mathsf{Aut}(X)$ is the group $\mathfrak{S}(G)$ of permutations of the set G, with the usual left Γ-operation on it, namely : $({}^{\sigma}\alpha)(x) = \rho(\sigma).\alpha(\rho(\sigma)^{-1}.x)$, for any $x \in G$, and where the dot "." denotes the product in G. The sub-sheaf $\mathsf{Aut}_G(X)$ of permutations α such that $\alpha(xg) = \alpha(x)g$ is clearly isomorphic, as a group, to G via the map $G \longrightarrow \mathsf{Aut}_G(X)$, $\quad a \mapsto (x \mapsto ax)$; the action of Γ on G transfered by this isomorphism is given by conjugations : ${}^{\sigma}g = \rho(\sigma).g.\rho(\sigma)^{-1}$. Looking at the adjoint group $\mathsf{Aut}_G(X)$ as a presheaf, its value on the Γ transitive object Γ/Γ' is

$$\mathrm{Hom}_\Gamma(\Gamma/\Gamma', \mathsf{Aut}_G(X)) = \mathrm{Aut}_G(X)^{\Gamma'} \simeq \mathrm{Cent}_G(\rho(\Gamma')),$$

that is the centralizer of $\rho(\Gamma')$ in G. Hence, in general, this adjoint group and G are not isomorphic, but their abelianizations are indeed isomorphic since the abelianization process neutralizes the operation of Γ by conjugation.

3.4.2 Now here is an example of two locally isomorphic objects of $B\Gamma$ whose automorphism groups are easy to determine. Note first that any left Γ-set X is *locally* isomorphic to its underlying set X_0 endowed with the trivial action of Γ ; in fact one has the commutative triangle in $B\Gamma$

$$\begin{tikzcd} \Gamma \times X_0 \arrow[rr, "(\gamma,x)\mapsto(\gamma,\gamma x)"] \arrow[dr, "\mathrm{pr}_1"'] & & \Gamma \times X \arrow[dl, "\mathrm{pr}_1"] \\ & \Gamma & \end{tikzcd}$$

whose horizontal arrow is an isomorphism of Γ-sets, and $\Gamma \longrightarrow e$ is an epimorphism.
Let T be a finite set endowed with two left Γ-actions given by two group homomorphisms

$$u : \Gamma \longrightarrow \mathfrak{S}(T), \qquad v : \Gamma \longrightarrow \mathfrak{S}(T).$$

We note $T(u)$ and $T(v)$ the Γ-sets they define. As seen above, the group $\mathsf{Aut}_{\mathsf{E}}(T(u))$ is nothing but the symmetric group $\mathfrak{S}(T)$ endowed with the action of Γ given by

$$^{\sigma}\alpha = u(\sigma) \circ \alpha \circ u(\sigma)^{-1}.$$

If u and v are not conjugate in $\mathfrak{S}(T)$, the actions of Γ on $\mathsf{Aut}_{\mathsf{E}}(T(u))$ and on $\mathsf{Aut}_{\mathsf{E}}(T(v))$ are not isomorphic, but the objects $T(u)$ and $T(v)$ are locally isomorphic (to the constant object T), so the corollary gives isomorphisms

$$\mathsf{Aut}_{\mathsf{E}}(T(u))^{\mathrm{ab}} = (\mathfrak{S}(T))^{\mathrm{ab}} \ \simeq \ \{\pm 1\} \ \simeq (\mathfrak{S}(T))^{\mathrm{ab}} = \mathsf{Aut}_{\mathsf{E}}(T(v))^{\mathrm{ab}}$$

which are anyway perfectly well known.

Remarks
3.5.1 It is perhaps worth recalling an evidence : two locally isomorphic groups may have non isomorphic abelianizations ; for example if they are already abelian, and not isomorphic ! It is the case for the groups μ_n and $\mathbf{Z}/n\mathbf{Z}$ in the topos of left $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$-sets.
3.5.2 If being concerned with a greater generality, we should have introduced the whole lower central series of the groups H and G, and something like the graded Lie algebra associated with ; then, we should have noticed a whole sequence of isomorphisms of groups $\mathrm{gr}^n(H) \simeq \mathrm{gr}^n(G)$, extending the case $n = 1$ of the theorem : $\mathrm{gr}^1(H) = H^{\mathrm{ab}} \simeq G^{\mathrm{ab}} = \mathrm{gr}^1(G)$. But we will deal later with G-object X which are no more torsors, and thus with adjoint groups which are semi-direct products, and Daniel CONDUCHÉ kindly put me off some very naïve attemps at calculating the Lie algebra of such groups. So more general statements will wait.
3.5.3 Given a group G, when dealing with "central functions" on G, viz. with characters, one often needs to introduce the set $G_\natural$ of conjugacy classes of elements of G ; it is usually not a group, nor the map $G_\natural \to G^{\mathrm{ab}}$ be bijective. The proof of the theorem shows that a (H, G)-bitorsor gives rise to an isomorphism

$$\theta_X^\natural : H_\natural \longrightarrow G_\natural.$$

It obviously contains much more "information" than the group isomorphism $\alpha_X : H^{\mathrm{ab}} \longrightarrow G^{\mathrm{ab}}$, but the lack of usual structures on the set of conjugacy classes makes these "informations" difficult to grasp.

4. Abelianization of groupoïds

The most obvious generalization of the notion of group is the notion of *groupoïd* : it is a category whose all arrows are isomorphisms. For any object x of such a category, the set $\mathrm{Hom}(x, x)$ is in fact a group, called the *stabilizer* of x, and denoted by $\mathrm{Aut}(x)$.
A groupoïd is said to be *abelian* if all its stabilizers $\mathrm{Aut}(x)$ are abelian groups ([Gi], IV 2.2.3.4). This paragraph explains how to associate to any groupoïd G (in a topos) its abelianization G^{ab}, and it shows that an abelian (transitive) groupoïd in the above GIRAUD sense do have the properties of the BREEN definition : roughly speaking, the (abelian) groups $\mathrm{Aut}(x)$ are "canonically" isomorphic.

The abelianized groupoïd G^{ab} has the same set of objects as G ; the definition of the arrows of G^{ab} rests on the following remark. For two objects x and y of G, the set $\mathrm{Hom}(x, y)$, if not empty, is a

$(\mathrm{Aut}(y), \mathrm{Aut}(x))$ bitorsor. Therefore, to define the abelianization G^{ab} it is not enough to pass to the quotients $\mathrm{Aut}(x) \longrightarrow \mathrm{Aut}(x)^{\mathrm{ab}}$; we must also replace the sets $\mathrm{Hom}(x, y)$ by their quotients obtained by pulling them along the above morphisms, in order to get $(\mathrm{Aut}(y)^{\mathrm{ab}}, \mathrm{Aut}(x)^{\mathrm{ab}})$ bitorsors ; more explicitly, let $D(x) \subset \mathrm{Aut}(x)$ be the subgroup generated by the commutators ; then we let

$$\mathrm{Hom}_{G^{\mathrm{ab}}}(x, y) = D(y)\backslash\mathrm{Hom}_G(x, y) = \mathrm{Hom}_G(x, y)/D(x).$$

Let G be a transitive groupoïd in **Set** (transitive means that all the $\mathrm{Hom}(x, y)$ are non empty) ; choose a pinning ("épinglage") of G ; it consists of the choice of an object $z \in X$, together with the choice, for each $x \in X$, of an arrow $u_x : x \to z$; we then dispose of a bijection

$$G \longrightarrow \mathrm{Aut}(z) \times X \times X, \quad (u : x \to y) \longmapsto (u_y \circ u \circ u_x^{-1}, x, y)$$

In the opposite direction the map is given by $(w, x, y) \mapsto u_x \circ w \circ u_y^{-1} \in \mathrm{Hom}(x, y)$. If the group $\mathrm{Aut}(z)$ is abelian it is clear that the isomorphism $\mathrm{Aut}(x) \to \mathrm{Aut}(z)$, $u \mapsto u_x \circ u \circ u_x^{-1}$ is independent from the choice of u_x. It is also clear that, in general, one has a bijection $\mathrm{Hom}_{G^{\mathrm{ab}}}(x, y) \simeq \mathrm{Aut}(z)^{\mathrm{ab}} \times \{x\} \times \{y\}$. Needless to say that if $G = \Gamma \times X$ is the groupoïd associated with a transitive action of a group Γ on X, then its abelianization has nothing to do with $\Gamma^{\mathrm{ab}} \times X$; besides Γ^{ab} does not act on X.

These considerations are now extended to a groupoïd in a topos E. We shall follow the presentation by Deligne, [De], **10.2 - 10.8**.
A (transitive) groupoïd over an object X of E is a covering

$$G \longrightarrow X \times X$$

endowed with a "composition law" (where the p_{ij} are the projections $X \times X \times X \longrightarrow X \times X$)

$$\circ : \mathrm{p}_{12}^\star G \times \mathrm{p}_{23}^\star G \longrightarrow \mathrm{p}_{13}^\star G \tag{4.1}$$

with the following properties : for any S in E, the set $G(S)$ has the properties of the set of arrows of a groupoïd with set of objects $X(S)$, the composition of arrows being given by the above composition law. The stabilizer St_G is the pull-back of G along the diagonal :

$$\begin{tikzcd} \mathrm{St} \arrow[r] \arrow[d] & G \arrow[d] \\ X \arrow[r] & X \times X \end{tikzcd}$$

It is a group over X, and G is a $(p_1^\star\mathrm{St}, p_2^\star\mathrm{St})$-bitorsor ([De] **10.8**, or, more explicitely, [U 1], Thm 2.1). Repeating what was said in the beginning for groupoïds in **Sets**, we can now define

$$G^{\mathrm{ab}} \;=\; p_1^\star\mathrm{St}^{\mathrm{ab}} \overset{p_1^\star\mathrm{St}}{\wedge} G = G \overset{p_2^\star\mathrm{St}}{\wedge} p_2^\star\mathrm{St}^{\mathrm{ab}}$$

The second equality perhaps needs a comment : let $D \subset \mathrm{St}_G$ be the sheaf of subgroups generated by the commutators ; that is the kernel of $\mathrm{St} \to \mathrm{St}^{\mathrm{ab}}$; we must define a canonical isomorphism

$$p_1^\star D\backslash G \xrightarrow{\sim} G/p_2^\star D. \tag{4.2}$$

First note the commutativity of the following diagram where products are over $X \times X$, and where the horizontal arrows are isomorphisms

$$\begin{tikzcd} p_1^\star\mathrm{St} \times G \arrow[r] \arrow[rr, bend left, "\theta"] \arrow[dr, "\mathrm{pr}_2"'] & G \times G \arrow[d, "\mathrm{pr}_2"] & G \times p_2^\star\mathrm{St} \arrow[l] \arrow[dl, "\mathrm{pr}_1"] \\ & G & \end{tikzcd}$$

Since θ is an isomorphism of groups over G it induces an isomorphism between the commutators subgroups : $p_1^\star D \times G \;\xrightarrow{\sim}\; G \times p_2^\star D$; in terms of sections, denoting by (g, d_2) the image of (d_1, g) by this isomorphism, one has

$$d_1 g \;=\; g\, d_2.$$

In other words, θ induces an isomorphism of the the left action of $p_1^\star D$ on G with the right action of $p_2^\star D$ on G. Hence the isomorphism (4.2).

Let consider now an *abelian* transitive groupoïd $G \longrightarrow X \times X$; denote by $A = \mathrm{St}_G$ its stabilizer ; it is an abelian group over X. We will show that A descends to the final object e, i.e that there exists a group A_0 in E, and an isomorphism of groups over X, $X \times A_0 \xrightarrow{\sim} A$, with compatibility conditions which will be specified.
Since, by assumption, the map $X \to e$ is a covering, every descent datum for A over X is effective (A.3.2). But the theorem **3.2** shows that the $(p_1^\star A, p_2^\star A)$-bitorsor G gives rise to an isomorphism

$$\alpha_G : p_1^\star A \xrightarrow{\sim} p_2^\star A.$$

It is a descent datum for A : in fact, the "composition law" (4.1) induces, following ([De], **10.8**), an isomorphism of $(\mathrm{pr}_1^\star A, \mathrm{pr}_3^\star A)$-bitorsors over the product $X \times X \times X$ (with its three projections on X denoted by pr_i)

$$\mathrm{p}_{12}^\star G \wedge^{\mathrm{pr}_2^\star A} \mathrm{p}_{23}^\star G \xrightarrow{\psi} \mathrm{p}_{13}^\star G$$

Since α maps bitorsor isomorphisms to identity, and since it transforms wedge products into compositions, one has :

$$\mathrm{p}_{23}^\star(\alpha_G) \circ \mathrm{p}_{12}^\star(\alpha_G) = \alpha_{\mathrm{p}_{23}^\star G} \circ \alpha_{\mathrm{p}_{12}^\star G} = \alpha_{\mathrm{p}_{12}^\star G \wedge \mathrm{p}_{23}^\star G} = \alpha_{\mathrm{p}_{13}^\star G} = \mathrm{p}_{13}^\star(\alpha_G)$$

Hence α_G is indeed a descent datum for A. The descent property shows that there exists a group A_0 and an isomorphism of groups over X,

$$\varphi : X \times A_0 \longrightarrow A$$

such that the following square is commutative

$$\begin{tikzcd}
X \times X \times A_0 \arrow[r, "p_1^\star\varphi"] \arrow[d, equal] & p_1^\star A \arrow[d, "\alpha_G"] \\
X \times X \times A_0 \arrow[r, "p_2^\star\varphi"] & p_2^\star A
\end{tikzcd}$$

In particular, with the definition (2.9) of [Br 2], G is thus an "abelian A_0-groupoïd".

5 Free operations

5.1 Definition *Let G be a group in a topos E. A right G-object X in E is G-free if the morphism*

(5.1.1) $$X \times G \longrightarrow X \times X, \quad (x,g) \longmapsto (x, xg)$$

is a monomorphism in E.

Given a set I, the coproduct $I \times G = \coprod_I G$ is G-free ; more generally, the coproduct $\coprod_{i \in I} X_i$ of a family $(X_i)_{i \in I}$ of G-torsor is also G-free. Another important example of a G-free object is given by a group X in E and a monomorphism of groups $G \to X$, the (right) action of G on X being given by the product on the right.

5.2 The monomorphism (5.1.1) defines an equivalence relation on X ; by (**A.4**) the quotient exists, denoted by

$$\pi : X \longrightarrow X/G,$$

and this morphism is *effective universal* (SGA 4, I 10.10, p. 184) ; writing $Q = X/G$, this means that the sequence

$$X \times G \overset{(x,g)\mapsto xg}{\underset{\mathrm{pr}_X}{\rightrightarrows}} X \xrightarrow{\pi} Q$$

is exact in the topos E, and that the morphism

(5.2.1) $$X \times G \longrightarrow X \times_Q X, \quad (x,g) \longmapsto (x, xg)$$

is an isomorphism ; furthermore, these properties remain true after any base change $Q' \to Q$.

The following consequence will often be used : given two maps $T \overset{x}{\underset{y}{\rightrightarrows}} X$ such that $\pi x = \pi y$, that is a map $T \to X \times_Q X$, there exists a unique $g : T \to G$ such that $y = xg$.

For more details on free actions, see [Gi], III 3.1.

In general, for a G-object X we do not dispose of subobjects of X like orbits ; we will show that strong finiteness conditions on X/G allow to overcome this difficulty by using the sheaf S of sections of π, which is then a covering of e. Recall that, for an object T of E, $S(T)$ is the set of maps $s : T \times (X/G) \longrightarrow X$ making the following triangle commutative

$$\begin{tikzcd} & X \arrow[d, "\pi"] \\ T \times (X/G) \arrow[ur, "s"] \arrow[r, "\mathrm{pr}_2"'] & X/G \end{tikzcd}$$

5.3 Lemma *Let X be an object in E, acted on freely by a group G. Denote by $\pi : X \longrightarrow X/G$ the quotient map, and by S the sheaf of sections of π.*
i) There is a map $\varphi : S \times X \longrightarrow G$ which is characterized, for any object T, any $s \in S(T)$ and $x \in X(T)$ by the relation

$$x = s(\pi x)\varphi(s, x)$$

ii) We have an isomorphism of right G-objects over S

$$S \times (X/G) \times G \xrightarrow{\sim} S \times X$$

given by $(s, \xi, g) \longmapsto (s, s(\xi)g)$.

i) Since $\pi \circ s = \mathrm{Id}$, x and $s(\pi x)$ have the same image in X/G, hence the existence of the the map $\varphi(s, x)$ to G satisfying the given relation. In terms of diagrams : let $\psi : S \times X \longrightarrow X$ be the composition $S \times X \xrightarrow{1\times\pi} S \times Q \xrightarrow{\text{can.}} X$, also written as $\psi(s, x) = s(\pi x)$. Then the map φ makes the following diagram commutative

$$\begin{tikzcd} X \times G \arrow[r, "(5.2.1)"] & X \times_Q X \\ & S \times X \arrow[ul, "(\psi,\varphi)"] \arrow[u, "(\psi,\mathrm{pr}_X)"'] \end{tikzcd}$$

ii) We merely describe the map in the other direction. It is $(s, x) \longmapsto (s, \pi(x), \varphi(s, x))$, where φ is the map introduced in *i)*.

5.4 Definition *An object Q in a topos E is said to be* locally constant finite[1] *if there is a finite set I, an epimorphism $Z \longrightarrow e$ and a Z-isomorphism*

$$Z \times Q \simeq Z \times I = \coprod_I Z$$

5.5 Proposition *Let X be an object acted on freely by a group G ; we suppose that X/G is locally constant finite.*
i) Let S be the sheaf (object in E) of sections of $\pi : X \to X/G$. Then $S \to e$ is a covering.
ii) The canonical map $\mathsf{Aut}_G(X) \times X \longrightarrow X \times X$ is a covering.

Write, as above, $Q = X/G$. Let $Z \to e$ be a covering such that $Z \times Q$ is constant.

1. In SGA 4, IX 2, the definition is slightly more general : instead of *one* epimorphism, it involves a covering family, and a set for each member of that family.

Proof of *i*). The commutativity of the square

$$\begin{tikzcd} Z \times S \arrow[r] \arrow[d] & S \arrow[d] \\ Z \arrow[r] & e \end{tikzcd}$$

shows that it is enough to prove that the map $Z \times S \longrightarrow Z$ is an epimorphism; therefore, we can assume that Q is constant, say $Q \simeq e \times I$. Each $i \in I$ gives rise to a monomorphism $e \to e \times I$, and thus to the subobject X_i of X, defined by pull back :

$$\begin{tikzcd} X \arrow[d] & X_i \arrow[l] \arrow[d] \\ e \times I & e \arrow[l, "i"'] \end{tikzcd}$$

One thus gets the coproduct splitting $X \xleftarrow{\sim} \coprod_{i\in I} X_i$, from which we deduce easily the following isomorphism in E

$$S \;\simeq\; \prod_{i\in I} X_i.$$

Since the quotient map $X \longrightarrow Q = X/G$ is an epimorphism, each pull back $X_i \longrightarrow e$ is also an epimorphism (**A.3**), and the same is true for their product $\prod_{i\in I} X_i \longrightarrow e$ since the set I is finite (epimorphisms are "universal").

For proving *ii*) we may suppose again that Q is constant finite, and that $X = \coprod_{i\in I} X_i$ is thus a finite coproduct of G-torsors. The map $\mathsf{Aut}_G(X) \times X \longrightarrow X \times X$ is defined by $(\alpha, x) \longmapsto (\alpha x, x)$; since product commute with finite coproduct (**A.2**), it is enough to check that, for all $i \in I$ the map

$$\mathsf{Aut}_G(X) \times X_i \;\longrightarrow\; X \times X_i$$

is an epimorphism; but by restricting an automorphism to X_i, we get the commutative square

$$\begin{tikzcd} \mathsf{Aut}_G(X) \times X_i \arrow[r] \arrow[d, "\text{restr.}"'] & X \times X_i \arrow[d, equal] \\ \mathsf{Hom}_G(X_i, X) \times X_i \arrow[r, "\text{ev.}"'] & X \times X_i \end{tikzcd}$$

The restriction map is an epimorphism since, in that case, it is nothing but the projection of a product onto one of its factor. So, it remains to show that the evaluation map is an isomorphism. But since X_i is a G-torsor, this map is locally of the form

$$\mathsf{Hom}_G(G_r, X) \times G_r \;\longrightarrow\; X \times G_r$$

Finally, one may invoke the "evident" isomorphism $X \simeq \mathsf{Hom}_G(G_r, X)$ induced by the map $X \times G \longrightarrow X$ ([Gi], III, 1.2.7 *i*)).□

So if the action of G is free then the action of $\mathsf{Aut}_G(X)$ is transitive; the symmetry shown in (2.3.1) is lost when X is not any longer a torsor.

6 The transfer

6.1 Theorem *Let G be a group in a topos E, and let X be an object in E endowed with a free (right) action of G. We suppose that the quotient X/G is locally constant finite and is not a terminal object. Then there is an isomorphism of groups in E*

$$\mathsf{Aut}_G(X)^{\mathrm{ab}} \;\xrightarrow{\sim}\; G^{\mathrm{ab}} \times \{\pm 1\}. \tag{6.1.1}$$

If $X/G \simeq e$, then X is a G-torsor (2.2); so X is a $(\mathsf{Aut}_G(X), G)$-bitorsor and we have already discussed in (3.2) the existence and the properties of the isomorphism $\alpha_X : \mathsf{Aut}_G(X)^{\mathrm{ab}} \xrightarrow{\sim} G^{\mathrm{ab}}$.

In general, the group morphism Ver : $\mathsf{Aut}_G(X)^{\mathrm{ab}} \longrightarrow G^{\mathrm{ab}}$ got by ignoring the factor ± 1 deserves the name of *transfer* ; see (6.4) for the link with the usual notion.

Proof of (6.1). Let denote by S the object in E representing the sheaf of sections of π. By (5.5), the map $S \to e$ is a covering. By (5.3), the G-object X is split up over S as the coproduct of trivial G-torsors :

$$S \times (X/G) \times G \ \xrightarrow{\sim}\ S \times X$$

The objects $(X/G) \times G$ and X being locally isomorphic, the corollary (3.3) gives an isomorphism

$$\mathsf{Aut}_G(X)^{\mathrm{ab}} \xrightarrow{\sim} \mathsf{Aut}_G((X/G) \times G)^{\mathrm{ab}}.$$

6.2 Lemma *Let Z be an object, and G be a group. Then one has an isomorphism of groups in* E

$$\mathsf{Aut}_G(Z \times G) \xrightarrow{\sim} \mathsf{Hom}(Z, G) \rtimes \mathsf{Aut}(Z),$$

where the semi-direct product on the right is associated with the left action of $\mathsf{Aut}(Z)$ *on* $\mathsf{Hom}(Z, G)$ *given by* ${}^{\alpha}m = m \circ \alpha^{-1}$

The isomorphism $Z \simeq (Z \times G)/G$ gives a split epimorphism $\mathsf{Aut}_G(Z \times G) \to \mathsf{Aut}(Z)$. Let f be a G-automorphism of $Z \times G$, and let α be the image of f, that is the composite

$$Z = Z \times e \longrightarrow Z \times G \xrightarrow{f} Z \times G \xrightarrow{\mathrm{pr}_Z} Z$$

Let now $m : Z \longrightarrow G$ be the map

$$Z \xrightarrow{\alpha^{-1}} Z \longrightarrow Z \times G \xrightarrow{f} Z \times G \xrightarrow{\mathrm{pr}_G} G$$

With these notations, we may write f as

(6.2.1) $$(z, g) \ \longmapsto\ (\alpha(z), m(\alpha(z))g).$$

It is then trivial to check that the map given by $f \mapsto (m, \alpha)$ is a group isomorphism. □

Thus the above result reduces the proof of (6.1) to establishing an isomorphism

(6.2.2) $$(\mathsf{Hom}(X/G, G) \rtimes \mathsf{Aut}(X/G))^{\mathrm{ab}} \ \simeq\ G^{\mathrm{ab}} \times \pm 1$$

6.3 First consider the case of sets.
Let Q be a *finite* set with $\mathrm{Card}(Q) \geq 2$, and let G be a group. Let $\mathrm{Hom}(Q, G)$ denote the group of all maps (of sets) $m : Q \to G$. Since Q is finite, the canonical map $\mathrm{Hom}(Q, G)^{\mathrm{ab}} \longrightarrow \mathrm{Hom}(Q, G^{\mathrm{ab}})$ is an isomorphism. The abelianization of a semi-direct product is well-known : we get the following group isomorphism

$$(\mathrm{Hom}(Q, G) \rtimes \mathfrak{S}(Q))^{\mathrm{ab}} \ \simeq\ \mathrm{Hom}(Q, G^{\mathrm{ab}})_{\mathfrak{S}(Q)} \times \mathfrak{S}(Q)^{\mathrm{ab}}$$

where the subscript $\mathfrak{S}(Q)$ denotes the quotient by the subgroup generated by the maps of the form $m^{-1}.m \circ \sigma$, with $\sigma \in \mathfrak{S}(Q)$.
Let $A = G^{\mathrm{ab}}$. We now use the following classical fact. The *trace* $\mathrm{Hom}(Q, A) \longrightarrow A,\ m \longmapsto \prod_{q \in Q} m(q)$ (which is well defined since Q is finite and A is abelian), induces an isomorphism

$$\mathrm{Hom}(Q, A)_{\mathfrak{S}_Q} \xrightarrow{\sim} A$$

This map being clearly surjective it is enough to check its injectivity. Denote by $N \subset \mathrm{Hom}(Q, A)$ the subgroup generated by the maps of the form $m^{-1}.m \circ \sigma$, with $\sigma \in \mathfrak{S}_Q$. Let $m : Q \to A$ be a map such that $\prod_{q \in Q} m(q) = 1$; to show that $m \in N$, we argue by induction on the support $\mathrm{Supp}(m) \subset Q$ of m. If q is in the support, there is an other element $q' \in \mathrm{Supp}(m)$ since the trace of m is trivial. Let n be the map defined by $n(q) = m(q)^{-1}$, $n(q') = m(q)$ and $n(q'') = 1$ for $q'' \neq q, q'$. Then n is clearly in the subgroup N and the support of $n.m$ is strictly contained in $\mathrm{Supp}(m)$.□

Finally, since $\mathrm{Card}(Q) \geq 2$, the signature gives an isomorphism $\mathfrak{S}(Q)^{\mathrm{ab}} \simeq \{\pm 1\}$, and we have got an isomorphism

$$(\mathrm{Hom}(Q,G) \rtimes \mathfrak{S}_Q)^{\mathrm{ab}} \ \simeq\ G^{\mathrm{ab}} \times \{\pm 1\}$$

That ends the proof of the set-theoretical form of (6.1).

6.4 Still restrict ourself to sets ($\mathsf{E} = \mathbf{Sets}$).
Here we explicit the transfer by following the steps involved in the above proof. Choose a section $s : Q = X/G \to X$ of the projection π. Let u be a G-automorphisme of X, and let $\bar{u}$ be the bijection it induces on Q (with the notations of (6.2), one has $\bar{u} = \alpha$). By (5.3), we have a G-isomorphism $Q \times G \to X, \quad (q,g) \mapsto s(q)g$, and (6.2) introduces the map $m : Q \to G$ such that

(6.4.1) $$u(s(q)) \ =\ s(\bar{u}(q)).m(\bar{u}(q)).$$

The transfer is defined in (6.3) as

$$\mathrm{Ver}(u) \ =\ \prod_{q \in Q} m(q)^{\mathrm{ab}}.$$

If instead of the section s we use its associated "representative subset" $T = \mathrm{Im}(s)$ as in the introduction, we define the bijection $\sigma \in \mathfrak{S}(T)$ by denoting $\sigma(t)$ the "representative" of $u(t)$; with the notations above we have $\sigma(t) = s(\bar{u}(\bar{t}))$, and the element $g_t \in G$ from the introduction is $m(\bar{t})$; finally, the relation (6.4.1) becomes the relation

(6.4.2) $$u(t) = \sigma(t) g_{\sigma t}$$

used in the introduction.

When X is a group containing G as a subgroup (of finite index), and when the map u is given by the product on the left by an element of X, still denoted by u, the relation (6.4.2) may be written as a product in the group X : $u.t = \sigma(t).g_{\sigma t}$, or $g_{\sigma t} = \sigma(t)^{-1}.u.t$, and thus

$$\mathrm{Ver}(u) \ =\ \prod_{t \in T} (\sigma(t)^{-1}.u.t)^{\mathrm{ab}}.$$

It is the usual formulation of the transfer in group theory.

6.5 We now return to the proof of the existence of the isomorphism (6.2.2) in the frame of a general topos E.

$$(\mathsf{Hom}(X/G, G) \rtimes \mathsf{Aut}(X/G))^{\mathrm{ab}} \ \simeq\ G^{\mathrm{ab}} \times \pm 1.$$

The proof is essentially the same as above once we dispose of a *trace map*

$$\mathsf{Hom}_{\mathsf{E}}(Q,G) \ \longrightarrow\ G^{\mathrm{ab}},$$

that is a map of functors in T,

$$\mathrm{Hom}(T, \mathsf{Hom}_{\mathsf{E}}(Q,G)) \ \longrightarrow\ \mathrm{Hom}(T, G^{\mathrm{ab}})$$

Restricting to the topos E/T of objects over T, we can boil down to the case where T is the terminal object. By assumption, there exist a covering $Z \to e$ and an isomorphism $u : \coprod_I Z \simeq Z \times Q$, where I is a finite set. We have a sequence of bijections

(6.5.1) $$\mathrm{Hom}(Z, \mathsf{Hom}_{\mathsf{E}}(Q,G)) \simeq \mathrm{Hom}(Z \times Q, G) \simeq \mathrm{Hom}(\coprod_I Z, G) \simeq \prod_I \mathrm{Hom}(Z,G)$$

Hence, taking the product, we get the trace map

$$\mathrm{Hom}(Z, \mathsf{Hom}_{\mathsf{E}}(Q,G)) \ \longrightarrow\ \mathrm{Hom}(Z,G)^{\mathrm{ab}} \ \longrightarrow\ \mathrm{Hom}(Z, G^{\mathrm{ab}})$$

We have to check that this map decends from Z to e. Let u_1 and u_2 be the two inverse images over $Z \times Z$ of the isomorphism $u : \coprod_I Z \simeq Z \times Q$

$$\textstyle\coprod_I Z \times Z \underset{u_2}{\overset{u_1}{\rightrightarrows}} \ Z \times Z \times Q$$

They differ by a permutation $\sigma \in \mathfrak{S}(I)$ of the factors : $u_2 = u_1 \circ \sigma$. Consider now the bijection (6.5.1) $v : \mathrm{Hom}(Z, \mathsf{Hom}_{\mathsf{E}}(Q,G)) \longrightarrow \prod_I \mathrm{Hom}(Z,G)$; its inverse images

$$\mathrm{Hom}(Z\times Z, \mathsf{Hom}_{\mathsf{E}}(Q,G)) \underset{v_2}{\overset{v_1}{\rightrightarrows}} \textstyle\prod_I \mathrm{Hom}(Z\times Z, G)$$

differ by the same permutation σ of the factors as u_1 and u_2 do ; the products defining the traces are equal, since they take place in the commutative group $\mathrm{Hom}(Z\times Z, G^{\mathrm{ab}})$. But $Z \to e$ is effective universal, therefore the trace over Z comes from a trace over e (A.3.2).

Once the trace map is shown to exist we can copy the combinatorial argument used above to conclude that it gives an isomorphism in E

$$\mathsf{Hom}(X/G,\, G^{\mathrm{ab}})_{\mathsf{Aut}(X/G)} \xrightarrow{\sim} G^{\mathrm{ab}}.$$

The theorem is proven. □

7. The transfer as determinant

We retain the hypotheses and the notations of the §6 : X is acted on freely by the group G, and X/G is locally constant finite. Let $D(G)$ be the commutator subgroup of G. The abelian group $G^{\mathrm{ab}} = G/D(G)$ acts freely on $X/D(G)$; according to (6.1), the canonical morphism $\mathsf{Aut}_G(X) \longrightarrow \mathsf{Aut}_{G^{\mathrm{ab}}}(X/D(G))$ induces an isomorphism

$$\mathsf{Aut}_G(X)^{\mathrm{ab}} \quad \xrightarrow{\sim} \quad \mathsf{Aut}_{G^{\mathrm{ab}}}(X/D(G))^{\mathrm{ab}} \quad (\simeq G^{\mathrm{ab}} \times \{\pm 1\})$$

This remark justifies the hypothesis we add now : *the group G is abelian.*

Still denote by $\mathbf{Z}$ the object of the topos E associated to the constant presheaf $\mathbf{Z}$. Given an object X of E, we denote by

$$\mathbf{Z}X$$

the *free abelian group* generated by X ([Gr-V], 11.3.3, p.500), also written $\mathbf{Z}_X$ or $\mathbf{Z}^{(X)}$; it represents the sheaf associated to the presheaf $T \mapsto \mathbf{Z}^{(\mathrm{Hom}(T,X))}$, and there is a bijection, functorial in the abelian group M of E,

$$\mathrm{Hom}_{\mathsf{Ab}(\mathsf{E})}(\mathbf{Z}X, M) \quad \xrightarrow{\sim} \quad \mathrm{Hom}_{\mathsf{E}}(X, M).$$

Since G is an abelian group, $\mathbf{Z}G$ is a commutative ring of E, and a G-object X gives rise to a $\mathbf{Z}G$-module $\mathbf{Z}X$. Since X is G-free and X/G is locally isomorphic to the constant set $I \times e$, with $\mathrm{Card}(I) = n$, the $\mathbf{Z}G$-module $\mathbf{Z}X$ is locally free of rank n ; in fact, according to (5.3) and (5.5), $\mathbf{Z}X$ is locally isomorphic to $\mathbf{Z}(I\times G) = \prod_I \mathbf{Z}G \simeq (\mathbf{Z}G)^n$. The module $\wedge^n \mathbf{Z}X$ (wedge power as $\mathbf{Z}G$-module), is thus locally isomorphic to $\mathbf{Z}G$. Hence the determinant is well defined in this context, and it gives a "concrete" (?) group homomorphism

$$\mathsf{Aut}_G(X) \xrightarrow{u \mapsto \mathbf{Z}u} \mathsf{Aut}_{\mathbf{Z}G}(\mathbf{Z}X) \xrightarrow{\wedge^n} \mathsf{Aut}_{\mathbf{Z}G}(\wedge^n \mathbf{Z}X) \simeq (\mathbf{Z}G)^{\times}$$

7.1 Proposition *Let G be an* abelian *group in E, and let X be a free G-object such that X/G is locally constant finite. Then the determinant map*

$$\det : \mathsf{Aut}_G(X) \quad \longrightarrow \quad (\mathbf{Z}G)^{\times}$$

takes its values into the subgroup $G \times \{\pm 1\} \subset (\mathbf{Z}G)^{\times}$, and it is equal to the map of (6.1) : transfer $\times$ signature :

$$\det(\mathbf{Z}u) = \mathrm{Ver}(u).\mathrm{sgn}(\bar u).$$

In other words, the following square is commutative

$$\begin{tikzcd}
\mathsf{Aut}_G(X)^{\mathrm{ab}} \arrow[r, "\mathrm{can.}"] \arrow[d, "\mathrm{Ver}\times\mathrm{sgn}"'] & \mathsf{Aut}_{\mathbf{Z}G}(\mathbf{Z}X)^{\mathrm{ab}} \arrow[d, "\det"] \\
G \times \{\pm 1\} \arrow[r] & (\mathbf{Z}G)^{\times}
\end{tikzcd}$$

(For the left vertical arrow be an isomorphism, we must suppose that the signature is surjective, i.e. that X is not a G-torsor).

Verifying the statement can be made locally ; so we may suppose by (5.3) and (5.5) that X is the disjoint union $I \times G = \coprod_{i \in I} \{i\} \times G$ of copies of G, where I is a finite set ; the elements $(i, 1) \in I \times G$ form a $\mathbf{Z}G$-basis of $\mathbf{Z}X$; according to (6.2) and (6.2.1), to any G-automorphism u of X is associated a permutation α of I, and a map $m : I \to G$; the $\mathbf{Z}G$-linear map $\mathbf{Z}u$ induced by u sends the basis element $(i, 1)$ onto $(\alpha(i), m(\alpha i))$; it is the composition of the permutation α, whose determinant is $\mathrm{sign}(\alpha) \in \{\pm 1\}$, and of the linear map given on the basis by $(j, 1) \mapsto (j, m(j)) = (j, 1).m(j)$, whose determinant is $\prod_j m(j) \in G$.□

8. Extensions of groups

The transfer as such is not used in this paragraph but the methods previously introduced are shown to apply also to group extensions.

Here we are dealing with sets.

8.1 Embedding of a group extension in a wreath-product.

This construction goes probably back to I. Schur ; anyway, when the group G is abelian it is nowadays well known and it can be found, for example, in the exercises 7 and 8 of [A] I, §6, p.148-149. We don't assume G to be abelian before (8.5).

Consider a group extension

(8.1.1)
$$1 \longrightarrow G \longrightarrow E \xrightarrow{\pi} Q \longrightarrow 1$$

The image $\pi(x) \in Q$ of an element $x \in E$ will often be denoted by $\bar{x}$. We write E_r for the right G-set associated to the product of elements of G on the right.
We have the exact sequence of groups

(8.1.2)
$$1 \longrightarrow \mathrm{Hom}(Q, G) \longrightarrow \mathrm{Aut}_G(E_r) \longrightarrow \mathrm{Aut}(Q)$$

where, in $\mathrm{Hom}(Q, G) = \mathrm{Hom}_{\mathrm{Sets}}(Q, G) = \prod_Q G$ and $\mathrm{Aut}(Q) = \mathrm{Aut}_{\mathrm{Sets}}(Q)$, we ignore the group structure on Q ; nevertheless, we will use the group structure on $\mathrm{Hom}(Q, G)$ coming from that of G, and we consider the group structure on $\mathrm{Aut}(Q)$ given by the composition of bijections. In (8.1.2), the left arrow sends a map $m : Q \to G$ to the automorphism $x \mapsto m(\bar{x})x$ of the right G-set E_r .
Let us check the exactness of the sequence. Let $u : E \to E$ be a G-map ; then $x \mapsto u(x)x^{-1}$ is constant on the G-orbits, and hence it defines a map $m : E/G = Q \longrightarrow E$; if u induces the identity on E/G, then $\pi(u(x)x^{-1}) = 1$, so the values of m are in G.

Consider now the homomorphisms $\mathrm{mult}_E : E \longrightarrow \mathrm{Aut}_G(E)$ and $\mathrm{mult}_Q : Q \longrightarrow \mathrm{Aut}(Q)$ given by multiplication on the left, and the exact sequence induced by pull-back along mult_Q :

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{Hom}(Q, G) \arrow[r] \arrow[d, equal] & E' \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d, "\mathrm{mult}_Q"] & 1 \\
1 \arrow[r] & \mathrm{Hom}(Q, G) \arrow[r] & \mathrm{Aut}_G(E_r) \arrow[r] & \mathrm{Aut}(Q) &
\end{tikzcd}$$

The initial extension (8.1.1) and the one just obtained fit into the following diagram :

(8.1.3)
$$\begin{tikzcd}
1 \arrow[r] & G \arrow[r] \arrow[d, "\mathrm{diag}"'] & E \arrow[r, "\pi"] \arrow[d, "\lambda"] & Q \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \mathrm{Hom}(Q, G) \arrow[r] & E' \arrow[r, "\pi'"'] & Q \arrow[r] & 1
\end{tikzcd}$$

By definition, the group E' is the set of maps $u : E \to E$ such that $u(xg) = u(x)g$ for $x \in E$ and $g \in G$, and whose induced map $\bar{u} : Q \to Q$ is the left multiplication by some $q \in Q$; the group law is given by the composition of maps.

Let $s : Q \to E$ be a set-theoretic section of π. As in (5.3) we denote by $\varphi(s, x)$ the element of G such that

$$x \;=\; s(\bar{x})\varphi(s, x)$$

Note that $\varphi(s, xg) = \varphi(s, x)g$, and that $\varphi(s, 1) = 1$, if $s(1) = 1$. We define a map

$$s_{\text{grp}} : Q \;\longrightarrow\; E', \qquad s_{\text{grp}}(q) = (x \mapsto s(q\bar{x})\varphi(s, x))$$

It is clear that $\pi' \circ s_{\text{grp}} = \mathrm{Id}_Q$; the main point is that s_{grp} is in fact a group-theoretic section of π', as one can check immediately.

8.2 Proposition *Under the hypotheses above, and with the same notations, the map*

$$s \;\longmapsto\; s_{\text{grp}}$$

defines a bijection between the set of set-theoretic sections of π such that $s(1) = 1$, and the set of group-theoretic sections of π'.
A set-theoretic section s (with $s(1) = 1$) is a group theoretic section if and only if $s_{\text{grp}} = \lambda \circ s$

The proof is straightforward as long as the map in the opposite direction is known ; so, let $t : Q \to E'$ be a group-theoretic section of π' ; then $t(q)$ has to be seen as a map $E \to E$, and the associated set-theoretic section of π is $q \mapsto t(q)(1)$.
The condition $s(1) = 1$ is needed for getting a bijection. In any case, we can replace a set-theoretic section s by $s.s(1)^{-1}$ in order to fulfill this condition.$\square$

A group-theoretice section $t : Q \longrightarrow E'$ of π' gives an isomorphism of groups between the wreath-product $\mathrm{Hom}(Q, G) \rtimes Q$ and E' ; whence the title

8.3 Under the hypotheses of (8.1), let S be the set of set-theoretical sections of π. We define a left operation of E' on S by letting, for $u \in E' \subset \mathrm{Aut}_G(E_r)$, ${}^u s = u \circ s \circ \bar{u}^{-1}$, as shown from the diagram

$$\begin{array}{ccc} E & \xrightarrow{u} & E \\ \uparrow{\scriptstyle s} & & \uparrow{\scriptstyle {}^u s} \\ Q & \xrightarrow[\bar{u}]{} & Q \end{array}$$

When restricted to E, this operation becomes ${}^{\lambda(x)}s = (q \mapsto xs(\bar{x}^{-1}q))$, and when restricted to a map u coming from $\mathrm{Hom}(Q, G)$, i.e. $u(x) = m(\bar{x})x$, the operation reads as $({}^u s)(q) = m(q)s(q)$.

Restricting this action of E' to the subgroup $\mathrm{Hom}(Q, G) \subset E'$ makes S a left $\mathrm{Hom}(Q, G)$-torsor. This simple observation combined with the Frattini argument, recalled below, shows that π' is split, as already shown in (8.2).

8.4 (The Frattini argument) *Let F be a subgroup of a group E, and let X be a (left) E-set. If the restriction to F of the operations of E makes X a F-torsor, then for every element $x \in X$ with stabilizer E_x, one has $F \cap E_x = 1$, and $F.E_x = E$. If, moreover F is normal in E, we get a decomposition of E as a semi-direct product $E = F \rtimes E_x$.*$\square$

8.5 We will now show how these considerations and a trivial case of the *méthode de la trace*[2] give a proof of the (abelian case of the) Schur-Zassenhauss theorem :

Suppose that Q is a finite group, of order say N, suppose that G is abelian and that the homomorphism $g \mapsto g^N$ is an automorphism of G. Then the sequence (8.1.1) is split.

Since G is abelian, we dispose of a trace map $\mathsf{tr} : \mathrm{Hom}(Q, G) \;\longrightarrow\; G$, and its restriction to the "diagonal" subgroup $G \subset \mathrm{Hom}(Q, G)$ is the automorphism $g \mapsto g^N$.
We use the notations of (8.1), and the diagram (8.1.3). Since G and $\mathrm{Hom}(Q, G)$ are normal subgroups of E and E' respectively, there is an operation by conjugation of Q on both of them. Let us check that the trace

2. [SGA 4], IX, §5.

$\mathsf{tr} : \mathrm{Hom}(Q,G) \longrightarrow G$ is in fact a Q-morphism : let $q \in Q$, and let $a \in E$ be a lifting of q ; the operation of q on $g \in G$ is ${}^q g = aga^{-1}$; the map $(x \mapsto ax) \in E'$ is a lifting of q ; therefore, for $m \in \mathrm{Hom}(Q,G)$, one has $({}^q m)(q') = {}^q(m(q^{-1}q')) = am(q^{-1}q')a^{-1}$; it is then clear that $\mathsf{tr}({}^q m) = {}^q(\mathsf{tr}(m))$. Let $H = \mathrm{Ker}(\mathsf{tr})$; since tr is a Q-morphism, H is normal in E'. The group $F = E'/H$ contains $G = \mathrm{Hom}(Q,G)/H$ as a subgroup, and the quotient F/G is isomorphic to Q.

Let again S be the set of set-theoretic sections of π ; as seen in (8.3), it is a E'-left object, and a $\mathrm{Hom}(Q,G)$-torsor ; pushing it along tr we get

$$X \;=\; H\backslash S \;=\; G \wedge^{\mathrm{Hom}(Q,G)} S.$$

It is a F-left object and a G-torsor. These constructions are summarized in the following commutative diagram

$$\begin{tikzcd}
 & G \arrow[r] \arrow[d, "\delta"'] & E \arrow[r] \arrow[d, "\lambda"] & Q \arrow[d, equal] \\
S \arrow[dd] & \mathrm{Hom}(Q,G) \arrow[r] \arrow[d, "\mathsf{tr}"'] & E' \arrow[r] \arrow[d, "\mu"] & Q \arrow[d, equal] \\
 & G \arrow[r] & F \arrow[r] & Q \\
X & & &
\end{tikzcd}$$

The middle vertical map $\mu \circ \lambda$ is an isomorphism since the left vertical map $G \to G$ is the isomorphism $g \mapsto g^N$. As X is an F-left object, and a G-torsor, the Frattini argument shows that F decomposes as a semi-direct product $G \rtimes F_x$, where x is a point of X ; in other words, the lower horizontal sequence is split ; by isomorphism, the upper horizontal sequence is also split.□

A. Terminology from topos theory

Here is a list of the properties we needed in the text. They mainly come from [Gi], 0, §1 and §2. We confess a naïve point of view : we do not care about universes.

A.1 Limits are representable in E (limits corresponds to the french *limites projectives*) ; in particular, E has a final object denoted by e_{E} or e, and products exist in E([SGA 4], II 4.1, 3) ; [Gi] 0.2.7).

A.2 All set-indexed sum (alias coproduct) $\coprod_{i\in I} X_i$ exist in E, and are disjoint and stable by pullbacks : $(\coprod_{i\in I} X_i) \times Y \simeq \coprod_{i\in I}(X_i \times Y)$

A.3 All epimorphisms are effective and universal. We often use the more intuitive word of *covering* as a synonym of epimorphism.
Explicitly, if $f : X' \longrightarrow X$ is an epimorphism, then

A.3.1 The sequence $X' \times_X X' \overset{f_1}{\underset{f_2}{\rightrightarrows}} X' \xrightarrow{f} X$ is exact and it remains exact after any base change $Y \longrightarrow X$.

A.3.2 Every descent datum on objects or maps over X', and relative to f is effective ([SGA 3], IV 2.3)

A.3.3 Let $u : Y \to Z$ be a map of objects over X (i.e. a map in the induced topos E/X), and let u' be the corresponding map over X'. Then u is an epimorphism (resp. a monomorphism, resp. an isomorphism) if the same is true for u'.

A.4 Equivalence relations in E are effective and universal ([Gi], 0 2.7) Effectiveness of an equivalence relation

$$R \overset{u_0}{\underset{u_1}{\rightrightarrows}} X$$

means two things :

i) There exists a morphism $u : X \longrightarrow X/R$ which is a coequalizer of (u_0, u_1) ; explicitly, for any object Z, the following sequence is exact

$$\mathrm{Hom}(X/R, Z) \longrightarrow \mathrm{Hom}(X,Z) \rightrightarrows \mathrm{Hom}(R,Z)\ .$$

In particular, u is an epimorphism.

ii) The canonical morphism

$$R \longrightarrow X \times_Y X$$

is an isomorphism ([Gi] 0, 2.6.2)

A.5 In some places we use the word "associated sheaf" ; it refers to the canonical topology on E. A sheaf for this topology is representable ([Gr-V], 1.2) ; the representative of the associated sheaf may be given a description without any reference to a topology, but as a limit :

Let $F : \mathsf{E}^o \longrightarrow \mathsf{Ens}$ be a contravariant functor, and let X be a representative of the associated sheaf $\mathbf{a}F$. Then there is a map of presheaves $\varphi : F \longrightarrow h_X$ which is "universal" in the following sense : for any object Y and any map of presheaves $\psi : F \longrightarrow h_Y$ there exists a unique morphism $\theta : X \longrightarrow Y$ in E making commutative the triangle

$$\begin{tikzcd} F \arrow[rr, "\varphi"] \arrow[dr, "\psi"'] & & h_X \arrow[dl, "h_\theta"] \\ & h_Y & \end{tikzcd}$$

(In fact, since h_Y is a sheaf for the canonical topology one has $\mathbf{a}h_Y = h_Y$, and the map ψ induces a map $\mathbf{a}\psi : \mathbf{a}F = h_X \longrightarrow \mathbf{a}h_Y = h_Y$.)

Bibliography

[A] N. Bourbaki, *Algebra I*, Springer-Verlag (1989)

[Br 1] L. Breen, Bitorseurs et cohomologie non abélienne, in *The Grothendieck Festschrift*, vol. I, Birkhäuser, (1990)

[Br 2] L. Breen, *On the classification of 2-gerbes and 2-stacks*, Astérisque 225 (1994)

[Ca] P. Cartier, Sur une généralisation du transfert en théorie des groupes, *L'Enseignement mathématique,*, t.XVI, fasc.1, p.49-57, (1970)

[De] P. Deligne, Le groupe fondamental de la droite projective moins trois points, in *Galois Groups over* **Q**, Ihara, Ribet, Serre Eds, Springer-Verlag 1989

[Gi] J. Giraud, *Cohomologie non abélienne*, Springer-Verlag, (1971)

[Gr-V] A. Grothendieck, J.-L. Verdier, Topos, *Exposé IV de* [SGA 4].

[S 1] J.-P. Serre, *Cohomologie galoisienne*, Cinquième édition, LNM $n°5$, Springer-Verlag, (1994)

[S 2] J.-P. Serre, *Goupes finis*, `arXiv:math.GR/0503154 v5`, 22 Jul 2005

[U 1] K.-H. Ulbrich, On the correspondence between gerbes and bouquets, *Math.Proc. Cambridge Philos. Soc.*, **108**, (1990)

[U 2] K.-H. Ulbrich, On cocycles bitorsors and gerbes over a Grothendieck topos, *Math.Proc. Cambridge Philos. Soc.*, **110**, (1991)

[SGA 3] M. Demazure, A. Grothendieck, *Schémas en groupes*, I, LNM $n°151$, (1970) ; an edited and TeX...ed version is also available online at `www.math.ens.fr/~ gille`.

[SGA 4] M. Artin, A. Grothendieck, J.-L. Verdier, *Théorie des topos et cohomologie étale des schémas,*I, LNM $n°269$, Springer-Verlag, (1972)