SET THEORETICAL COMPLETE INTERSECTIONS IN CHARACTERISTIC $p > 0$

by Daniel Ferrand.

The main motivation for the remarks presented here is to clarify (for me) some mysterious calculations Hartshorne made in [2] : he showed that the rational curves $C_d \subset \mathbb{P}^3_k$, where $\mathrm{char}(k) = p > 0$, defined parametrically by

$$(u,v) \longrightarrow (u^d, u^{d-1}v, uv^{d-1}, v^d)$$

are the set-theoretical intersection of two surfaces, the equations of which he produced. It seems to me that one way to understand these calculations - i.e., finally, to avoid them - is indicated in the following more general result:

Let C be a smooth curve in $\mathbb{P}^3_k$, where k is algebraically closed and $\mathrm{char}(k) = p > 0$. Assume that C has a linear projection birational to it, and with only cusps as singularities. Then C is a set-theoretical complete intersection.

The proof may be summed up as follows: Let X be the cone in $\mathbb{P}^3_k$ associated to the given projection, and $f:\overline{X} \to X$ its normalization. There exists an effective divisor $\overline{C}$ on $\overline{X}$ mapped isomorphically onto C by f (section 1) . As f is radicial, the Frobenius morphism brings $\overline{C}$ back to an effective divisor on X , whose lifting Y to the ambient space is a surface that $X \cap Y = C$ (as sets).

Unfortunately I do not know how restrictive the hypothesis is; it is at least satisfied for the curves C_d, because C_d is of degree d and has a tangent line with a contact of order $d-1$.

Section 3 contains a criterion for a finite morphism to be the composite of a radicial morphism and an unramified morphism. It is used in Section 4 where we give a straightforward proof of a beautiful result by Cowsik and Nori.

1. The divisor associated with a curve on a cone

The result of this section was the basic tool for early geometers (Cayley, Halphen...); they stated it as: "A curve in space is the partial intersection of a cone and a monoid." Its translation into the language of cycles (theorem of Severi, [3], p. 98) has blunted it a little.

1.1. Fix an algebraically closed field k and a projective space $P = \mathbb{P}_k(V)$ built on a finite dimensional vector space V.

Let $x \in P$ be a closed point defined by a linear form $V \to k$, whose kernel will be denoted by V'. The linear projection with center x is the morphism

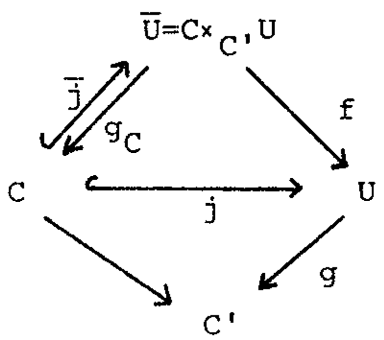
$$g: P - \{x\} \to P' = \mathbb{P}_k(V')$$

defined as follows: On $P - \{x\}$, the restriction to V'_P of the canonical quotient $\alpha: V_P \to \underline{O}_P(1)$ is still surjective, and g is the unique morphism such that this quotient is isomorphic to the pull-back $g^*(\alpha'): V'_{P'} \to g^*(\underline{O}_{P'}(1))$ of the canonical quotient on P'. Moreover, every splitting of the exact sequence $0 \to V' \to V \to k \to 0$ gives rise to an isomorphism of P'-schemes

$$\mathbb{V}_{P'}(\underline{O}_{P'}(-1)) \tilde{\to} P - \{x\}$$

<u>Proposition</u> 1.2 Let C be a smooth curve in P, not containing x, and such that the morphism $C \to g(C)$ induced by g is birational. Let X denote the cone over C with vertex x, and $f: \overline{X} \to X$ its normalization. Then there exists an effective divisor $\overline{C}$ on $\overline{X}$, defined by a section of $f^*(\underline{O}_X(1))$, and such that f induces an isomorphism $\overline{C} \tilde{\to} C$.

<u>Proof</u>: Let $j: C \to U = X - \{x\}$ denote the immersion of C in the punctured cone, and $C' = g(C)$ the projection of C on P'. Since U is isomorphic to $g^{-1}(C')$, the morphism $U \to C'$ is smooth, and the normalization $\overline{U}$ of U is isomorphic to $U \times_{C'} C$. Consider the following diagram, where $g_C \overline{j} = \text{id}$, and $f\overline{j} = j$:



Since $\overline{U}$ is isomorphic to $\mathbb{V}_C(\underline{O}_C(-1))$, $\overline{C} = \overline{j}(C)$ is an effective divisor on $\overline{U}$, defined by a section of $g^*(\underline{O}_C(1)) = f^*(\underline{O}_U(1))$; as $f\overline{j} = j$, f induces an isomorphism from $\overline{C}$ onto C. Now $\overline{U}$ is an open set in the normal scheme $\bar{X}$, and it contains all the points of codimension one in $\bar{X}$, so the section of $\underline{O}_{\bar{U}}(1)$ defining $\bar{C}$ extends to a section of $\underline{O}_{\bar{X}}(1)$.

1.3 The above straightforward construction is usually interpreted - and obscured - in terms of cycles and monoids, as follows:

Since $\bar{X} \to X$ is birational, the section of $\underline{O}_{\bar{X}}(1)$ defining $\bar{C}$ may be considered as a meromorphic section of $\underline{O}_X(1)$, and therefore it defines a (non effective) divisor D on X such that $f^*(D) = \bar{C}$.

Now let $\underline{Z}^1(X)$ denote the group of 1-codimensional cycles, and $\mathrm{cyc}:\mathrm{Div}(X) \to \underline{Z}^1(X)$ the cycle-map (EGA IV 21.6.7). The theorem of Severi alluded to above is essentially the following equality in $\underline{Z}^1(X)$:

$$\mathrm{cyc}(D) = C \ .$$

Let's make the monoids come into sight: Let N be any effective divisor on X containing the subscheme of X defined by the conductor of $\bar{X} \to X$. Then $M = D+N$ is an effective divisor on X , and this is a monoid. More precisely, N is the divisor associated to an invertible sub-$\underline{O}_X$-module $\underline{O}_X(N)$ of the field of rational functions on X , containing $\underline{O}_{\bar{X}}$. Therefore $\underline{O}_{\bar{X}}(1)$ is contained in $\underline{O}_X(1) \otimes \underline{O}_X(N)$, and the section of $\underline{O}_{\bar{X}}(1)$ associated to $\bar{C}$ furnishes a section of $\underline{O}_X(1) \otimes \underline{O}_X(N)$, which defines M . As a matter of facts, one usually chooses an N such that $\underline{O}_X(N) \cong \underline{O}_X(d)$, in order to be able to lift M to the ambient space (at least when $P = \mathbb{P}^3_k$), and it is the surface thus obtained which is called a monoid.

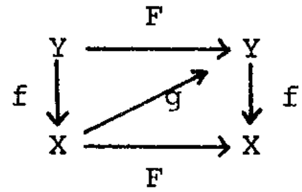
2. Application to complete intersections

2.1 A morphism of schemes $f:Y \to X$ is said to be radicial if it is injective and if for every $y \in Y$, the residual extension $k(f(y)) \to k(y)$ is radicial (hence trivial in characteristic 0).

The following criterion is useful: $f:Y \to X$ is radicial if and only if the diagonal morphism $\Delta_f:Y \to Y \times_X Y$ is surjective (EGA I 3.7.1).

Let C be an integral curve over an algebraically closed field, and let $f:\bar{C} \to C$ denote its normalization. Then f is radicial if and only if C has only cusps as singularities.

2.2 Let X be a scheme over the finite field $\mathbb{F}_p$. The Frobenius morphism $Fr: X \to X$ is radicial. Conversely, let $f: Y \to X$ be a finite radicial morphism such that $\underline{O}_X \to f_*(\underline{O}_Y)$ is injective, then there exists a power $q = p^n$ of the characteristic such that $f_*(\underline{O}_Y^q) \subset \underline{O}_X$. In other words, if $F = Fr^n$ denotes the n-th iterate of the Frobenius morphism, there exists a morphism $g: X \to Y$ making the following diagram commutative



Now, let L be an invertible $\underline{O}_X$-module, and $t: \underline{O}_Y \to f^*(L)$ a section on Y. Then we get a section <u>on X</u>,

$$g^*(t): \underline{O}_X \to g^*f^*(L) = F^*(L) = L^{\otimes q},$$

whose inverse image by f is $F^*(t) = t^q: \underline{O}_Y \to f^*(L^{\otimes q})$.

<u>Proposition</u> 2.3 Let C be a smooth connected curve in $\mathbb{P}_k^3$, where k is algebraically closed, and $char(k) = p > 0$. Assume the existence of a linear projection with center $x \notin C$, which maps C birationally onto a curve C' with only cusps as singularities. Then C is a set-theoretical complete intersection.

<u>Proof</u>: Keep the notations of 1.2. The morphism $f: \bar{X} \to X$ is radicial because $C \to C'$ is. By 2.2 , there exists a power $q = p^n$ of the characteristic, and an effective divisor D on X such that $f^*(D) = q\bar{C}$ and $\underline{O}_X(D) = \underline{O}_X(q)$. The underlying sets of C and D are the same. Moreover, any section of $\underline{O}_X(q)$ comes from a section of $\underline{O}_P(q)$ because X is a divisor in P . Therefore, there exists a surface Y in P such that $X \cap Y = D$.

<u>Remark</u> 2.4 This result is not known if $char(k) = 0$. The trick above shows at least that $X-C$ is affine: by a theorem of Chevalley (EGA II 6.7.1), this is equivalent to $f^{-1}(X-C)$ being affine; but $f^{-1}(X-C) = \bar{X}-\bar{C}$ because f is radicial, and $\bar{X}-\bar{C}$ is affine because $\bar{C}$ is ample.

3. A factorization for finite morphisms

Proposition 3.1 Let $f:Y \to X$ be a finite morphism between noetherian schemes.

The following conditions are equivalent:

1) There exists a factorization of f as $Y \overset{h}{\to} Z \overset{g}{\to} X$, where g is unramified, h is radicial, and $\underline{O}_Z \to h_*(\underline{O}_Y)$ is injective.

2) The underlying set of the diagonal in $Y \times_X Y$ is open.

Moreover, under these conditions, the factorization is unique.

Remarks 3.2.1 A morphism of finite type $f:Y \to X$ is unramified if and only if its diagonal morphism $\Delta_f:Y \to Y \times_X Y$ is an open immersion; this is, of course, a strictly stronger condition than 2). A finite morphism which is both radicial and unramified is a closed immersion. This implies the unicity in the above factorization.

3.2.2 If, in 3.1, X and Y are integral curves, and if f is birational, then the complement of the diagonal in $Y \times_X Y$ is a finite set of closed points, so the condition 2) is satisfied. If moreover Y is the normalization of X, then Z may be thought of approximately as the curve obtained from X by separating the branches at the multiple points.

Proof of 1 ⇒ 2): Consider the following commutative diagram where $u\Delta_h = \Delta_f$, and where the square is cartesian:

$$\begin{array}{ccccc} Y & \xrightarrow{\Delta_h} & Y \times_Z Y & \xrightarrow{u} & Y \times_X Y \\ & & \downarrow & & \downarrow{\scriptstyle h \times h} \\ & & Z & \xrightarrow{\Delta_g} & Z \times_X Z \end{array}$$

Since g is unramified, Δ_g is an open immersion, hence u is an open immersion. Since h is radicial, Δ_h is surjective.

Proof of 2) ⇒ 1): Write $A = \underline{O}_X$, and $B = f_*(\underline{O}_Y)$. We may - and we shall - suppose that $A \to B$ is injective.

To obtain Z, we shall iterate the following construction:

Let U denote the scheme induced by $Y \times_X Y$ on the open set $\Delta_f(Y)$. The direct image R of $\underline{O}_U$ on X is inserted between the following morphisms of A-algebras, where m is surjective and has a nilpotent kernel

$$B \rightrightarrows R \xrightarrow{m} B \ .$$

Take $C_1 = \mathrm{Ker}(B \rightrightarrows R)$, and $Z_1 = \mathrm{Spec}(C_1)$.

First we show that $h_1 : Y \to Z_1$ is radicial: The construction commutes with flat base change, so we can assume C_1 to be a strictly henselian local ring. Then, it suffices to prove that B is local. For every idempotent $e \in B$, we have, in $B \otimes_A B$,

$$(1-e\otimes 1-1\otimes e)(e\otimes 1-1\otimes e) = 0 \ .$$

The first factor is invertible in R bacause its image $1-2e$ is invertible in B, and $\mathrm{Ker}(m)$ is nilpotent. Therefore e is in the local ring C_1; so e is trivial.

This implies that h_1 induces a homeomorphism from Y onto Z_1; hence the condition 2) is satisfied for $g_1 : Z_1 \to X$, and we can iterate the construction.

It remains to prove that $Z_m \to X$ is unramified for m large. Let S_n denote the support of $\Omega^1_{Z_n/X}$ viewed as an A-module. We get a decreasing sequence of closed subsets of the noetherian space X; therefore, we have $S_n = S_{n+1} = \dots$, for n large enough, and we must show that S_n is empty. Suppose it is not, and let A' denote the strict henselisation of A at a maximal point of S_n. After replacing $B' = A' \otimes_A B$ by $C'_n = A' \otimes_A C_n$, we can assume $A' \to B'$ to be unramified outside the closed points. Let us show that the A'-module B'/A' is then artinian: Since A' is strictly henselian, B' is the direct product of local rings B'_i, and the residual extension of $A' \to B'_i$ is radicial; hence $B'_i \otimes_{A'} B'_i$ is a local ring. The underlying set of the diagonal $\mathrm{Spec}(B'_i) \subset \mathrm{Spec}(B'_i \otimes_{A'} B'_i)$ contains the unique closed point, and is open by condition 2). Therefore it is the whole space, and the morphism $\mathrm{Spec}(B'_i) \to \mathrm{Spec}(A')$ is radicial. But this morphism is unramified outside the closed point, hence it is an isomorphism there, and B'/A' is artinian. This implies that the decreasing sequence C'_m/A' of sub-A'-modules of B'/A' is stationary, hence $C'_m = C'_{m+1} = \dots$, for m large enough. If $x \in C'_m$, the very construction of C'_{m+1} from C'_m shows that the support of $x\otimes 1-1\otimes x$ in $\mathrm{Spec}(C'_m \otimes_{A'} C'_m)$ is disjoint from the diagonal. Therefore the morphism $\mathrm{Spec}(C'_m) \to \mathrm{Spec}(A')$ is unramified. This contradiction completes the proof.

3.3 This result will be applied, in the next section, to a birational morphism between reduced curves. In this case, the proof is shorter because the A-module B/A is obviously artinian.

4. The theorem of Cowsik-Nori

The following result is the main point in this paper [1] by Cowsik and Nori. Their proof seems a little intricate; here I shall give a proof which is more geometrical and, I hope, easier to follow.

Proposition 4.1 Let k be an algebraically closed field of characteristic $p > 0$, and C a reduced curve in $P = \mathbb{P}^3_k$, with any singularities. Then there exists a curve D in P which is locally a complete intersection, such that $D_{red} = C$.

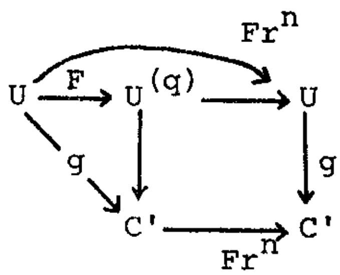
Proof: Choose a linear projection $g: P - \{x\} \to P'$, with center $x \notin C$, which induces a birational morphism from C onto its image C'. Consider the factorization given by 3.1

$$C \xrightarrow{u} C'' \xrightarrow{v} C' ,$$

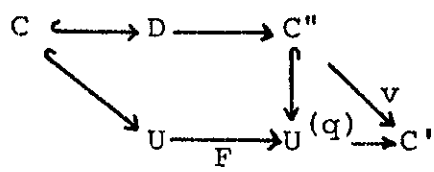
where u is radicial and v unramified. The exists a power $q = p^n$ of the characteristic such that, with slight abuses of notation, one has

4.1.1 $$\underline{O}_{C''} = \underline{O}_{C'}[\underline{O}_C^q]$$

Call $U = g^{-1}(C')$ the punctured cone over C, and $F: U \to U^{(q)}$ the n-th power of the relative Frobenius morphism: by definition, one has the following commutative diagram with cartesian square



The equality 4.1.1 above means that C'' is the schematic image $F(C)$ of the subscheme C of U. Let $D = F^{-1}F(C) = U \times_{U^{(q)}} C''$. One gets the following factorization of the morphism $C \to C'$:



Since C' is a reduced plane curve, $\Omega^1_{C'/k}$ is locally generated by two elements. The exact sequence

$$v^*(\Omega^1_{C'/k}) \to \Omega^1_{C''/k} \to \Omega^1_{C''/C'} \to 0$$

and the fact that v is unramified imply that $\Omega^1_{C''/k}$ is also locally generated by two elements; therefore C'' is locally a complete intersection.
Now $F:U=\mathbb{V}_{C'}(\underline{O}_{C'}(-1)) \to U^{(q)}=\mathbb{V}_{C'}(\underline{O}_{C'}(-q))$ is easily seen to be a complete intersection morphism (EGA IV 19.3); this implies that D is locally a complete intersection. Finally $u:C \to C''$ and $F:D \to C''$ are surjective homeomorphisms, hence $D_{red} = C$.

Bibliography

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