Prime Ideals in the Burnside Ring

Daniel Ferrand

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This note is an attempt to clarify the remarkable concise paper by A. Dress, entitled *A characterisation of Solvable Groups* [D 69], where he gives a description of the spectrum of the Burnside ring which allows him to prove that this spectrum is connected if and only if the group is solvable.

The Burnside ring Burn(G) of a finite group G is the Grothendieck ring of finite G-sets. A set T with a transitive action of G may be identified, after the choice of an element $t \in T$, with the quotient set G/H, where H is the stabilizer of the point t. But the passage from the subgroup H to the quotient $T = G/H$ is, in a sense, contravariant ; because it is repeatedly used by Dress, it may get the reader, like me, lost in a kind of counter-intuitive hesitation. So I chose, in this paper, to use the language of G-sets alone (except for some specific proofs). Another obvious motive for doing so is a view toward a future study of a "Burnside ring" for the finite étale coverings of a given scheme (hopefully more accessible than the Burnside ring of the fundamental group of the scheme).

In what follows, G denotes a *finite* group, and all the G-sets are finite.
These restrictions will no more be specified.

1. Connected G-sets

A (finite) set T endowed with a transitive action of G will be called *connected* G-sets [1]. By definition, a connected G-set is non empty. We usually reserve the letters T, U, S for the connected G-sets. The transitivity property is equivalent to the map

$$G \times T \longrightarrow T \times T, \qquad (g, t) \longmapsto (gt, t)$$

being surjective.

For two G-sets X and Y, we will use the term *morphism* to indicate a map $f : X \longrightarrow Y$ of G-sets, that is a map such that $f(gx) = gf(x)$ for all $g \in G$ and all $x \in X$.

A morphism $X \longrightarrow T$ where X is non empty and T connected, is surjective ; a surjective morphism $X \longrightarrow T$ whose domain X is connected has necessarily a connected target T.

The following fact will be repeatedly used :

Two morphisms of opposite direction, $T \to U$ and $U \to T$, between connected sets are necessarily isomorphisms.

2. Pointing

The pointing of a connected set T is the choice of an element $t \in T$. Such a choice allows us to introduce the stabilizer $G_t = \{g \in G, gt = t\}$ of t, and the isomorphism of G-sets

$$G/G_t \xrightarrow{\sim} T. \tag{2.1}$$

This association between a connected set and a subgroup may sometimes help in offering intuition of these things. But,if, instead of sets, we consider sheaves, or objects in a topos, the pointing process may be impossible, at least globally ; moreover, the choice of an element, when possible, is mostly arbitrary, and different choices lead to different isomorphisms ; for example, the stabilizer of an other element of T, written as gt, is the conjugate subgroup : $G_{gt} = gG_tg^{-1}$. For these reasons, we try, in this note, to avoid the pointing process at least in defining/constructing objects ; but we are less reluctant to use it when it seems that some proofs are easier for a quotient G/H ; it can then be looked upon as a kind of localization.

1. Bourbaki uses the adjective *homogeneous* [A, I 5.5.]

3. Degree

Let $f : X \longrightarrow Y$ be a morphism. The degree of f, denoted by $\deg(f)$, is the map $Y \to \mathbf{N}$ defined by $y \longmapsto \mathrm{Card}(f^{-1}(y))$.
A morphism is surjective if its degree map is nowhere zero.
If Y is connected, the degree map is constant since $f^{-1}(gy) = gf^{-1}(y)$.

3.1. Lemma *Let $f : X \longrightarrow Y$ be a morphism. Then f is of constant degree d if and only if there exists a surjective morphism $Y' \longrightarrow Y$ such that $X \times_Y Y'$ be the disjoint union of d copies de Y'.*

This strong analogy with the étale coverings from the algebraic geometry explains why a morphism f whose degree is constant, say $\deg f = d$, will often be called a *covering*, or even a *covering of degree d*

The proof uses an induction on the degree d, starting from the obvious case where $d = 1$. Suppose now that X be broke as the disjoint union $X = Y^{\sqcup m} \sqcup Z$, i.e. that f owns m G-sections with disjoint images [2]; if Z is non empty, its degree over Y is $d - m \neq 0$, thus the restriction of f to Z is a surjective morphism $f' : Z \to Y$; moreover, one has $X \times_{f,Y,f'} Z = Z^{\sqcup m} \sqcup Z \times_Y Z$, and the morphism $Z \times_Y Z \longrightarrow Z$ admits a section disjoint from the previous ones, the diagonal; then $f_Z : X \times_Y Z \longrightarrow Z$ admits at least $d+1$ sections with disjoint images.$\square$

4. Automorphisms

4.1 Let T be a connected G-set. Then the group $\mathsf{W}(T)$ of the G-automorphisms of T acts freely on T; in other words, the map

$$\mathsf{W}(T) \times T \longrightarrow T \times T$$

is injective; in fact, given two automorphisms w and w', if one element $t \in T$ has the same image $w(t) = w'(t)$, then for all $g \in G$, one has $w(gt) = gw(t) = gw'(t) = w'(gt)$, and thus $w = w'$, since the action of G is transitive.

For a morphism $f : X \to Y$, we denote by $\mathsf{W}(X/Y)$, or $\mathsf{W}(f)$, the group of the G-automorphisms w of X which are compatible with f, i.e $fw = f$.

The morphism f is said to be galoisian, *or a* Galois covering, *if it is surjective and if there exists a subgroup $\Gamma \subset \mathsf{W}(f)$ such that the map*

$$\Gamma \times X \longrightarrow X \times_Y X, \qquad (w, x) \mapsto (w(x), x)$$

is bijective, or equivalently, if the map $X/\Gamma \longrightarrow Y$ is bijective.

As usual, the group Γ is not uniquely determined by f if X is not connected; for a morphism between connected sets (hence f is surjective), things are easer :

4.2 Lemma *A morphism $f : T \to U$ between connected sets is galoisian if and only if*

$$\mathrm{Card}(\mathsf{W}(T/U)) \;=\; \deg(f).$$

Since f is surjective, the degree of $f_T = f \times 1_T : T \times_U T \longrightarrow T$ is equal to $\deg(f)$; from this we deduce the equality

$$\deg(f).\mathrm{Card}(T) \;=\; \mathrm{Card}(T \times_U T).$$

Hence the claim.$\square$

4.3. Lemma *Let Γ be a subgroup of $\mathsf{W}(T)$; then the quotient set T/Γ is a G-set, and the morphism*

$$f : T \;\longrightarrow\; T/\Gamma$$

is a Galois covering and one has $\Gamma = \mathsf{W}(f)$.

Since T is acted on freely by Γ, the orbits have all the same cardinality, namely the order d of Γ; these orbits are the fibers of the morphism $f : T \longrightarrow T/\Gamma = S$; thus f is a covering of degree d. The two morphisms

$$\Gamma \times T \longrightarrow \mathsf{W}(T/S) \times T \longrightarrow T \times_S T$$

2. If Y were connected, the images of distinct sections would necessarily be disjoint.

are injective. Since $T \to S$ is of degree d, the cardinality of $T \times_S T$ is equal to $d.\mathrm{Card}(T) = \mathrm{Card}(\Gamma \times T)$; therefore one has $\Gamma = \mathsf{W}(T/S)$, and f is a Galois covering.□

4.4. Lemma *Let S be connected, and let $f : X \longrightarrow S$ be a Galois covering with group W. Any connected component $T \subset X$ is galoisian over S with group the stabilizer of T in W.*

In fact, consider two elements $t, t' \in T$ in the same fiber. By definition of a Galois covering, there exists a unique $w \in W$ such that $w(t) = t'$. This automorphism w stabilizes T since every element in T may be written as gt, with $g \in G$, and one has $w(gt) = gw(t) = gt' \in T$.□

4.5. Remark. The result is no more true for a G-subset T of X which would not be connected. Consider for example a finite set I endowed with a cyclic permutation σ ; and let $X = S \times I$ the disjoint union of copies of S indexed by I ; let W be the group generated by the automorphism w given by $w(s, i) = (s, \sigma(i))$; the projection $X = S \times I \longrightarrow S$ is clearly a Galois covering with group W, since I is a torsor under the group $< \sigma >$. It is easy to find a specific set I containing a subset J whose stabilizer in $< \sigma >$ is reduced to the identity. Then the component $S \times J \subset S \times I$ can't be galoisian over S for a subgroup of W.

5. The case of pointed G-sets

Let $H \subset K$ be two subgroups of G, and let $f : G/H \longrightarrow G/K$ be the corresponding morphism. Its degree is the indice :

$$\deg(f) = (K : H).$$

The group of the G-automorphisms of the G-set G/H is isomorphic with $(\mathrm{Norm}_G(H)/H)^{\mathrm{o}}$ ([A] I 5.5, prop. 5) ; precisely, the (right) action of an $n \in G$ which normalizes H on the class gH is given by

$$(gH) \star n \;=\; gnH \;=\; gHn.$$

Thus the relative automorphisms, i.e. the elements of $\mathsf{W}(f)$, are given by the n which, moreover, fix the classes gK ; therefore

$$\mathsf{W}(f) = (\mathrm{Norm}_K(H)/H)^{\mathrm{o}}.$$

We deduce the following remark :

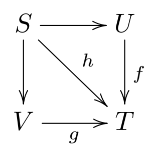
5.1. Lemma *The morphism $f : G/H \longrightarrow G/K$ is galoisian if and only if H is normal in K ; the Galois group is then $\mathsf{W}(f) = (K/H)^{\mathrm{o}}$.*

6. Existence of maximal p-galoisian coverings, and of maximal solvable coverings

I shall use the following definitions.

A covering is p-galoisian, also said a p-Galois covering, if it is a Galois covering whose Galois group is a p-group.
A solvable covering is a Galois covering whose Galois group is solvable.

6.1. Lemma *Let T, U, V be connected G-sets, and let $U \xrightarrow{f} T \xleftarrow{g} V$ be Galois coverings (resp. p-Galois coverings, resp. solvable coverings). Let $S \subset U \times_T V$ be a connected component of the fibered product. Then the composite morphism $h : S \longrightarrow T$ is galoisian (resp. etc.)*



To prove the lemma we choose a point $s \in S$, we denote by u, v, t its images, and we consider the stabilizers G_s, G_u, G_v, G_t ; since S is a subset of $U \times V$, one has $G_s = G_u \cap G_v$. According to (5.1), the hypothesis on f and g means that G_u and G_v are normal in G_t ; thus $G_u \cap G_v$ is also normal in G_t, i.e. h is galoisian. The group inclusion

$$G_t/G_u \cap G_v \ \subset \ G_t/G_u \ \times \ G_t/G_v$$

implies the more restrictive statements. □

6.2. Proposition *Let T be a connected G-set, and let p be a prime.*

i) Let $\varphi : T^{(p)} \longrightarrow T$ be a p-Galois covering for which $\mathrm{Card}(T^{(p)})$ is maximal. Then φ is also maximal in the "arrow" sense : namely, for any galoisian p-covering $\psi : U \longrightarrow T$, there exists a (non unique) morphism $\kappa : T^{(p)} \longrightarrow U$ such that $\varphi = \psi \circ \kappa$. In particular, this covering is unique up to a non unique isomorphism.

ii) Similarly, a maximal solvable covering $\sigma : T^{\mathrm{slv}} \longrightarrow T$ is also maximal in the above sense, and thus it is unique up to an isomorphism.

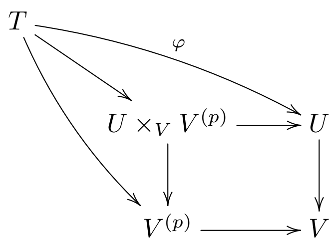
This is a direct consequence of the lemma above, since a subgroup of a solvable group is solvable.

6.3. Corollary *Any morphism $U \longrightarrow V$ between connected sets may be extended as a morphism $U^{(p)} \longrightarrow V^{(p)}$ of their maximal p-Galois coverings, and also as a morphism $U^{\mathrm{slv}} \longrightarrow V^{\mathrm{slv}}$.*

If the morphism $U \longrightarrow V$ is a p-Galois covering, then the composite $U^{(p)} \longrightarrow U \longrightarrow V$ is still a p-Galois covering, and thus $U^{(p)} \longrightarrow V^{(p)}$ is an isomorphism; moreover this construction is idempotent : $(U^{(p)})^{(p)} = U^{(p)}$.

The same is true for solvability : If the morphism $U \longrightarrow V$ is a solvable covering, then the composite $U^{\mathrm{slv}} \longrightarrow U \longrightarrow V$ is still a solvable covering, and thus $U^{\mathrm{slv}} \longrightarrow V^{\mathrm{slv}}$ is an isomorphism; moreover this construction is idempotent : $(U^{\mathrm{slv}})^{\mathrm{slv}} = U^{\mathrm{slv}}$.

Let P be the Galois group of $V^{(p)} \longrightarrow V$; by the base change we get a p-Galois covering $U \times_V V^{(p)} \longrightarrow U$ with the same group; let T be a connected component of $U \times_V V^{(p)}$; by (4.4), $\varphi : T \longrightarrow U$ is a Galois covering with group the stabilizer of T in P, that is a p-Galois covering.



From the proposition above the maximal p-galoisian covering $U^{(p)} \longrightarrow U$ factors through $\varphi : T \longrightarrow U$, and that gives the required morphism $U^{(p)} \longrightarrow V^{(p)}$.
Now suppose that the morphism $f : U \longrightarrow V$ be p-galoisian. To conclude that $f^{(p)} : U^{(p)} \longrightarrow V^{(p)}$ is an isomorphism, we have to show there exists a morphism in the other direction, and for doing so it is enough to show that the composite $U^{(p)} \longrightarrow U \longrightarrow V$ is a Galois covering. Choose a element $t \in U^{(p)}$ and denote by u and v its images. Considering the normalizers, we get the sequence of inclusions of normal subgroups

$$G_t \ \triangleleft \ G_u \ \triangleleft \ G_v$$

and we must check that G_t is normal in G_v. Since $U^{(p)} \to U$ is a maximal p-Galois covering, G_t is the minimal normal subgroup of G_u whose quotient is a p-group; its unicity implies that G_t is stable by any automorphism of G_u, and in particular, by any conjugation by an element of G_v.

The proof in the solvable case is formally the same.□

6.4. Corollary *Let T be a connected G-set, and let p be a prime. Suppose that there does not exist non trivial p-Galois covering $T' \to T$. Let $P \subset \mathsf{W}(T)$ be a p-Sylow subgroup of the automorphisms group of T. Then the morphism $(T/P)^{(p)} \longrightarrow T$ is an isomorphism.*

Since the morphism $T \longrightarrow T/P$ is a p-Galois covering (4.3), the maximal one $(T/P)^{(p)} \longrightarrow T/P$ factors trough T :

$$(T/P)^{(p)} \xrightarrow{\varphi} T \xrightarrow{\psi} T/P,$$

and φ is a p-Galois covering since the composite $\psi \circ \varphi$ is. The assumption on T implies that φ is an isomorphism.

7. Prime ideals in the Burnside ring.

The class of a (finite) G-set X in the Burnside ring $\mathrm{Burn}(G)$ will be noted $\mathsf{cl}(X)$.
Let T be a connected G-set ; then the set $\mathrm{Hom}_G(T, X)$ is finite, and its cardinality is denoted by $m_T(X)$, in homage to BURNSIDE who called this number the *mark* of T in X ([B], p.236) [3].

$$m_T(X) \;=\; \mathrm{Card}(\mathrm{Hom}_G(T, X)).$$

The marks extend to ring homomorphisms

$$m_T : \mathrm{Burn}(G) \;\longrightarrow\; \mathbf{Z}.$$

We choose a set $\mathcal{T}$ of connected G-sets containing exactly one element of each isomorphism classe. The marks together define a homomorphism

$$m : \mathrm{Burn}(G) \;\longrightarrow\; \mathsf{M}(\mathcal{T}, \mathbf{Z}) \quad x \longmapsto (T \mapsto m_T(x))$$

whose target denotes the ring of all maps from the set $\mathcal{T}$ to $\mathbf{Z}$.

7.1. Lemma *The set* $(\mathsf{cl}(T))_{T\in\mathcal{T}}$ *is a basis of* $\mathrm{Burn}(G)$ *and the morphism* m *is injective. In consequence,the ring* $\mathrm{Burn}(G)$ *is reduced.*

Since any finite G-set is the disjoint union of its orbits, the family $(\mathsf{cl}(T))_{T\in\mathcal{T}}$ generates the additive group $\mathrm{Burn}(G)$. An element $x \in \mathrm{Burn}(G)$ may thus be written as $x = \sum a_T \mathsf{cl}(T)$ with coefficients a_T in $\mathbf{Z}$. Suppose that some of these coefficients are non zero (it is the case if $x \neq 0$). Let $S \in \mathcal{T}$ be a connected set with minimal cardinality among those for which $a_T \neq 0$; since any morphism $S \to T$ is surjective, implying that $\mathrm{Card}(S) \geq \mathrm{Card}(T)$, we have $m_S(T) = 0$ for the other T, that is if $a_T \neq 0$ and $T \neq S$; therefore, one has $m_S(x) = a_S m_S(S) \neq 0$. That implies the two assertions. $\square$

7.2. With standard algebra arguments one deduces from this lemma a first characterization of minimal and maximal prime ideals of $\mathrm{Burn}(G)$, via their traces on the subring $\mathbf{Z} \subset \mathrm{Burn}(G)$; namely

7.2.1. A prime ideal $\mathfrak{p}$ is minimal if and only if $\mathbf{Z} \cap \mathfrak{p} = 0$.

7.2.2. A prime ideal $\mathfrak{m}$ is maximal if and only if $\mathbf{Z} \cap \mathfrak{m} = p\mathbf{Z}$, for a prime number p.

Consider a commutative algebra $K \to B$ of finite dimension over a field K ; then each prime ideal $\mathfrak{p}$ of B is maximal, and thus also minimal, since for x non zero in $B/\mathfrak{p}$, the K-linear map $B/\mathfrak{p} \to B/\mathfrak{p}$, $y \mapsto xy$ is injective, hence bijective.
This remark shows that the above conditions are sufficient. Let now $\mathfrak{p}$ be a minimal prime ideal in $B = \mathrm{Burn}(G)$. Since B is reduced, the local ring $B_{\mathfrak{p}}$ is a field, and the image of an element $a \in \mathbf{Z} \cap \mathfrak{p}$ in $B_{\mathfrak{p}}$ is zero. But, B being free over $\mathbf{Z}$, the composite map $\mathbf{Z} \to B \to B_{\mathfrak{p}}$ is flat ; thus, it sends a regular (i.e non zero) element of $\mathbf{Z}$ onto a regular element in $B_{\mathfrak{p}}$; hence $\mathbf{Z} \cap \mathfrak{p} = 0$.
Now let $\mathfrak{m}$ be a maximal ideal in B, we have to check that $\mathbf{Z}/\mathfrak{m} \cap \mathbf{Z}$ is a field. A non zero element a in that ring has a non zero, hence invertible, image in the field $B/\mathfrak{m}$; its inverse, $b = a^{-1}$ is the root of a monic polynomial $p(X)$, of degree say n, with coefficients in $\mathbf{Z}/\mathfrak{m} \cap \mathbf{Z}$. We thus have the relation $0 = a^n p(b) = 1 + ac$, with $c \in \mathbf{Z}/\mathfrak{m} \cap \mathbf{Z}$; it shows that a is invertible $\mathbf{Z}/\mathfrak{m} \cap \mathbf{Z}$.$\square$

The next theorem summarizes the description of $\mathrm{Spec}(\mathrm{Burn}(G))$, due to A. DRESS.

The number of automorphisms of a connected set T is denoted by $w(T) \;=\; m_T(T) \;=\; \mathrm{Card}(\mathsf{W}(T))$.

3. In fact Burnside identifies T with a quotient G/H, and speaks of the mark of H in the representation X.

7.3. Theorem *Let G be a finite group, and, as above, choose a set $\mathcal{T}$ of connected G-sets containing exactly one element from each isomorphism class.*

i) The assignment $\mathsf{cl}(T) \mapsto \mathfrak{p}_T = \mathrm{Ker}(\mathrm{Burn}(G) \xrightarrow{m_T} \mathbf{Z})$ defines a bijection between the set $\mathcal{T}$ and the set of minimal prime ideals of Burn(G).

ii) For $T \in \mathcal{T}$ and for a prime p, the kernel $\mathfrak{m}_{T,p}$ of the homomorphism

$$\mathrm{Burn}(G) \xrightarrow{m_T \,\mathrm{mod}.p} \mathbf{F}_p$$

is a maximal ideal. Moreover, the assignment $(\mathsf{cl}(T), p) \mapsto \mathfrak{m}_{T,p}$ defines a bijection *between the set of couples $(\mathsf{cl}(T), p)$ with $T \in \mathcal{T}$ and p a prime number not dividing $w(T)$, and the set of maximal ideals of* Burn(G).

iii) Let S and T in $\mathcal{T}$, and let p be a prime. Then the inclusion $\mathfrak{p}_S \subset \mathfrak{m}_{T,p}$ is equivalent to the congruence $m_S \equiv m_T \,\mathrm{mod}.p$.
If, moreover, the prime p does not divide $w(T)$, then the following conditions are equivalent
a) $m_S \equiv m_T \,\mathrm{mod}.p$.
b) There is a sequence of p-Galois coverings $S \to S_1 \to \cdots \to S_n = T$.
c) The maximal p-Galois coverings (6.2) of S and T have isomorphic domains : $S^{(p)} \xrightarrow{\sim} T^{(p)}$.
d) There exists a p-Galois covering $T^{(p)} \longrightarrow S$.

We begin with a key lemma.

7.4. Lemma *Let k be an integral domain, and let $f : \mathrm{Burn}(G) \longrightarrow k$ be a ring homomorphism. Let T be a connected set which is maximal (i.e. whose cardinality is maximal) among those such that $f(T) \neq 0$. Then f factors through m_T :*

$$\mathrm{Burn}(G) \xrightarrow[m_T]{} \mathbf{Z} \xrightarrow[\kappa]{} k\ , \qquad f = \kappa \circ m_T$$

where κ is the unique ring homomorphism $\mathbf{Z} \to k$.Two such connected sets are isomorphic.
Moreover, if κ is injective, then all the connected sets S such that $f = \varphi_S$ are isomorphic.

Let X be a G-set. We first compute $f(T \times X)$. It is the sum of the $f(U)$ for the connected components U of $T \times X$. By projection, such a component comes equipped with two maps

$$T \xleftarrow{\tau} U \xrightarrow{u} X$$

If τ is not an isomorphism, then the choice of T implies that $f(U) = 0$. If τ is an isomorphism, we associate with U the map $u\tau^{-1} : T \longrightarrow X$; conversely, to any map $v : T \longrightarrow X$ is associated its graph, which is a component of $T \times X$ isomorphic with T. These two constructions are clearly inverse each others, showing that the number of connected components of $T \times X$, which are isomorphic with T, is equal to $\mathrm{Card}(\mathrm{Hom}_G(T, X)) = m_T(X)$. We have thus shown that

$$f(T \times X) = m_T(X).f(T).$$

But, f being a ring homomorphism, one has $f(T \times X) = f(T).f(X)$. Since $f(T)$ is non zero in the domain k, we conclude that

$$f \ = \ \kappa \circ m_T.$$

Now, if T' is another connected set with $f(T') \neq 0$, then a fortiori the integer $m_T(T') = \mathrm{Card}(\mathrm{Hom}_G(T, T'))$ is not zero, i.e. there is a morphism $T \to T'$; by reversing the roles of T and T', we see that T and T' are isomorphic.

Suppose now that κ is injective, and thus f may be seen as a map to $\mathbf{Z}$, i.e. $f = m_T$. Let S be a connected set such that $m_S = m_T$; the integer $m_S(S) = \mathrm{Card}(W(S))$ is not zero, showing that there is a morphism $T \to S$; similarly, there is a morphism $S \to T$; thus S and T are isomorphic.□

7.5. Proof of the parts *i*) and *ii*) of the theorem.

Let $\mathfrak{p}$ be a prime ideal in Burn(G), and let $f : \mathrm{Burn}(G) \longrightarrow \mathrm{Burn}(G)/\mathfrak{p}$ be the quotient morphism. Consider a maximal connected set T such that $\mathsf{cl}(T) \notin \mathfrak{p}$; the above lemma implies that f factors through m_T, i.e. that $f = \kappa \circ m_T$; note that κ is surjective since f is. We thus have, for a G-set X,

(7.5.1) $$\mathsf{cl}(X) \ \equiv \ m_T(X) \,\mathrm{mod}.\mathfrak{p}.$$

For any T, the composite $\mathbf{Z} \xrightarrow{\text{can.}} \mathrm{Burn}(G) \xrightarrow{m_T} \mathbf{Z}$ is bijective. From that and (7.2.1), we deduce that the minimality of $\mathfrak{p}$ i.e. $\mathbf{Z} \cap \mathfrak{p} = 0$, is equivalent to the morphism $\kappa : \mathbf{Z} \longrightarrow \mathrm{Burn}(G)/\mathfrak{p}$ being injective (hence an isomorphism). The part $i)$ follows easily.

Consider now the case where κ is not injective. Since the morphism $\kappa : \mathbf{Z} \longrightarrow \mathrm{Burn}(G)/\mathfrak{p}$ is surjective one has $\mathrm{Burn}(G)/\mathfrak{p} = \mathbf{F}_p$ for a prime number p. The choice of T and the congruence (7.5.1) imply that

$$m_T(T) \not\equiv 0 \,\mathrm{mod}.p$$

In other words, the prime p does not divide the cardinality $w(T)$ of the group $\mathsf{W}(T)$. Together with (7.2.2), it is thus proven that each maximal ideal is of the form $\mathfrak{m}_{T,p}$ with $p \nmid w(T)$. It remains to check the injectivity of the asignment : if $m_T \equiv m_{T'} \,\mathrm{mod}.p$, then in particular, one has

$$m_T(T') \equiv m_{T'}(T') \not\equiv 0 \,\mathrm{mod}.p;$$

this implies the existence of a morphism $T \to T'$, and by symmetry, also a morphism $T' \to T$; hence T and T' are isomorphic.□

The following result is classical.

7.6. Lemma *Let E be a finite set acted on by a finite p-group P. Then, denoting by E^P the subset of invariants, one has*

$$\mathrm{Card}(E) \equiv \mathrm{Card}(E^P) \mod .p.$$

In particular, if $f : T \longrightarrow T/P$ is a p-Galois covering of G-sets, then $m_T \equiv m_{T/P} \mod .p$.

The congruence between the marks comes from the bijection

$$\mathrm{Hom}_G(T/P, X) \longrightarrow \mathrm{Hom}_G(T, X)^P, \quad u \longmapsto u \circ f.$$

□

7.7. Proof of the part $iii)$ of the theorem.

$\mathfrak{p}_S \subset \mathfrak{m}_{T,p} \Longleftrightarrow m_S \equiv m_T \,\mathrm{mod}.p$

The inclusion $\mathfrak{p}_S \subset \mathfrak{m}_{T,p}$ induces the canonical homomorphism $\mathrm{Burn}(G)/\mathfrak{p}_S \longrightarrow \mathrm{Burn}(G)/\mathfrak{m}_{T,p}$ which makes the following diagram commutative

$$\begin{array}{ccccccc}
\mathrm{Burn}(G) & \xrightarrow{\quad m_S \quad} & & & \mathbf{Z} & \xrightarrow{\sim} & \mathrm{Burn}(G)/\mathfrak{p}_S \\
\Big\| & & & & \downarrow & & \downarrow \\
\mathrm{Burn}(G) & \xrightarrow[m_T]{} & \mathbf{Z} & \longrightarrow & \mathbf{F}_p & \xrightarrow{\sim} & \mathrm{Burn}(G)/\mathfrak{m}_{T,p}
\end{array}$$

We thus have the congruence $m_S \equiv m_T \,\mathrm{mod}.p$. The converse is clear.

$a) \Rightarrow b)$.

Since $m_T(T) = w(T)$ is not a multiple of p, the congruence $m_S \equiv m_T \,\mathrm{mod}.p$ implies that $m_S(T) \not\equiv 0 \,\mathrm{mod}.p$, and, in particular, that there exists a morphism $S \to T$. If it is an isomorphism, there is nothing more to prove. If not, then $m_T(S) = 0$; but, by the property $a)$, one has $m_T(S) \equiv m_S(S) \mathrm{mod}.p$;thus p divides the order of $\mathsf{W}(S)$, and there is a non trivial p-Sylow subgroup P of $\mathsf{W}(S)$; it acts on the right on $\mathrm{Hom}_G(S, T)$. The lemma (7.6) above and the hypothesis $p \nmid w(T)$ imply the congruences

$$\mathrm{Card}(\mathrm{Hom}_G(S,T)^P) \overset{(7.6)}{\equiv} \mathrm{Card}(\mathrm{Hom}_G(S,T)) \overset{(a)}{\equiv} \mathrm{Card}(\mathrm{Hom}_G(T,T)) \not\equiv 0 \mod p$$

Therefore, there exists a morphism $f : S \longrightarrow T$ such that $fw = f$ for all $w \in P$; letting $S_1 = S/P$, the morphism f may now be factorized as

$$S \xrightarrow{f_1} S_1 \longrightarrow T,$$

where f_1 is a p-Galois covering, which is not trivial since $P \neq 1$. As $m_S \equiv m_{S_1} \mathrm{mod}.p$, the conclusion follows by induction on the degree of f.

$b) \Rightarrow c)$

This implication is proved in (6.3).

c) $\Rightarrow$ *d)*

By construction, the morphism $S^{(p)} \longrightarrow S$ is a p-Galois covering.

d) $\Rightarrow$ *a)*

The lemma (7.6) applied to the p-Galois coverings $T^{(p)} \longrightarrow T$ and $T^{(p)} \longrightarrow S$ gives the congruences

$$m_{T^{(p)}} \equiv m_T \equiv m_S \text{ mod.} p$$

□

8. Connected components of Spec(Burn(G)).

In this paragraph, the adjective *connected* will have two meanings, the second one being relative to Spec(Burn(G)) endowed with the Zariski topology, and its subsets. That may not cause any confusion.

As usual, for a prime ideal $\mathfrak{p}$ of Burn(G), we denote by $V(\mathfrak{p})$ the set of the primes which contain $\mathfrak{p}$; it is a closed connected set. In any topological space, if E and F are two connected sets such that $E \cap F \neq \emptyset$, then $E \cup F$ is connected. Thus a connected component C of Spec(Burn(G)) is a union of $V(\mathfrak{p})$ with $\mathfrak{p}$ a minimal prime ; in fact, it is a finite union since the ring Burn(G) has only a finite number of minimal prime ideals. Conversely, the sets E which may be written in the following manner are connected :

$$(\star) \qquad E = V(\mathfrak{p}_1) \cup V(\mathfrak{p}_2) \cup \cdots \cup V(\mathfrak{p}_n)$$

where the $\mathfrak{p}_i$ are minimal prime ideals, such that $V(\mathfrak{p}_i) \cap V(\mathfrak{p}_{i+1}) \neq \emptyset$, for $i = 1, 2, \cdots, n-1$. In this description we must allow some $\mathfrak{p}_i$ being equal : in fact, if $\mathfrak{p}$ is a minimal prime such that $V(\mathfrak{p}) \cap E \neq \emptyset$, but $V(\mathfrak{p}) \cap V(\mathfrak{p}_j) = \emptyset$ for $j \neq i$, then a sequence attached to the connected set $V(\mathfrak{p}) \cup E$ is

$$(\mathfrak{p}_1, \cdots, \mathfrak{p}_i, \mathfrak{p}, \mathfrak{p}_i, \cdots, \mathfrak{p}_n).$$

Now let E be a maximal set of the type described by $(\star)$. Let us show that E is a connected component of the spectrum. Since E is closed, it is enough to prove that E is open ; but the remark above shows that the complement of E is the union of the $V(\mathfrak{p})$ which are disjoint from E. Since, again, the ring Burn(G) has only a finite number of minimal prime ideals, this union is a closed subset.

8.1. Theorem *a) The minimal prime ideals $\mathfrak{p}_S$ and $\mathfrak{p}_T$ are in the same Zariski connected component of* Spec(Burn(G)) *if and only if there exits a connected G-set U and two* solvable *coverings*

$$S \overset{\sigma}{\longleftarrow} U \overset{\tau}{\longrightarrow} T,$$

or, equivalently, if the sources S^{slv} and T^{slv} of the maximal solvable coverings of S and T are isomorphic (cf. (6.3)).

b) For any non empty G-set U let $C(U) \subset$ Spec(Burn(G)) *be the union of the $V(\mathfrak{p}_T)$ for the various solvable coverings $U \to T$. Then $C(U)$ is connected (for the Zariski topology), and $C(U)$ is a connected component if and only if U has no non trivial abelian coverings, that is if the morphism $U^{\text{slv}} \longrightarrow U$ is an isomorphism.*

c) In particular, Spec(Burn(G)) *is connected if and only if the group G is solvable.*

a) For proving that the condition is sufficient, it is enough to show that $\mathfrak{p}_U$ and $\mathfrak{p}_T$ are in the same connected component. By choosing a Jordan-Hölder sequence of its (solvable) Galois group $\mathsf{W}(U/T)$, the solvable covering $\tau : U \longrightarrow T$ may be written as the composite of a sequence

$$U \overset{\tau_1}{\longrightarrow} U_1 \overset{\tau_2}{\longrightarrow} U_2 \to \cdots \to U_n = T,$$

where each $U_{i-1} \overset{\tau_i}{\longrightarrow} U_i$ is a cyclic Galois covering of order p_i for some prime (depending of i). Let

$$\mathfrak{p}_i = \text{Ker}(\text{Burn}(G) \overset{m_{U_i}}{\longrightarrow} \mathbf{Z})$$

be the minimal prime associated with U_i. The lemma (7.6) implies the following congruence between the marks

$$m_{U_{i-1}} \equiv m_{U_i} \,\mathrm{mod}.p_i$$

From the theorem (7.3 ; *iii*)) we deduce that the set $V(\mathfrak{p}_{i-1}) \cap V(\mathfrak{p}_i)$ contains the maximal ideal $\mathfrak{m}_{U_i,p_i}$. Hence $\mathfrak{p}_U$ and $\mathfrak{p}_T$ are in the same connected component of Spec(Burn(G)).
Conversely, suppose that the minimal primes $\mathfrak{p}_S$ and $\mathfrak{p}_T$ are in the same connected component, and that a sequence like ($\star$) above has been chosen with $\mathfrak{p}_1 = \mathfrak{p}_S$, and $\mathfrak{p}_n = \mathfrak{p}_T$; let S_i be a G-set corresponding with $\mathfrak{p}_i$. By assumption, a maximal ideal contains both $\mathfrak{p}_i$ and $\mathfrak{p}_{i+1}$; by (7.3 *iii*)), its is of the form $\mathfrak{m}_{U_i,p}$, for a suitable prime p, and there are two p-Galois coverings $S_i \longleftarrow U_i \longrightarrow S_{i+1}$; but a p-group being solvable, we see from (6.3) that the maximal solvable coverings S_i^{slv} and S_{i+1}^{slv} are isomorphic. Hence the conclusion.

b) The first part shows that a minimal prime $\mathfrak{p}_S$ is in the connected component containing $\mathfrak{p}_U$ if and only if S^{slv} and U^{slv} are isomorphic, that is if $\mathfrak{p}_S \in C(U^{\mathrm{slv}})$. The conclusion follows.

c) Consider the G-set reduced to one point ; it is canonically pointed with stabilizer equal to G, and thus it is isomorphic to G/G ; denoting by $D^\infty(G)$ the intersection of the derived series, the maximal solvable covering of the point is $G/D^\infty(G) \longrightarrow \bullet$. Let $\mathfrak{p}_\bullet$ be the prime ideal associated to the point ; if it is in the same component as the prime $\mathfrak{p}_G = \mathrm{Ker(Card)}$, then, from *a*) we deduce that the G-sets $G/D^\infty(G)$ and G are isomorphic, that is $D^\infty(G) = 1$, i.e. G is solvable. Conversely, if G is solvable, then for each subgroup H, the morphism $G \longrightarrow G/H$ is a Galois covering with group the solvable group H, showing that the spectrum is connected. □

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