CYCLIC HOMOLOGY OF SCHEMES AND SERRE'S L-FACTORS AS REGULARIZED DETERMINANTS

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Contents

1. Introduction

Cyclic homology and cyclic cohomology are two theories that had been introduced simultaneously by Connes and Tsygan at the beginning of the development of non-commutative geometry, in the years 1980. Tsygan worked more on the homology, while Connes on the other hand worked on the cohomology, introducing the periodic and negative cyclic cohomologies at the same time. The idea was to treat geometrically the study of non-commutative algebras, by extending Gel'fand duality between commutative C^*-algebras and topological spaces. From this new approach he developed special invariants which behave very well with the commutative and non-commutative cases, namely the cyclic homology. It can also be linked to algebraic K-theory through the *Chern-Connes character*. The definition of this new homology theory with Connes' (b, B)-bicomplex shows that this is also strongly related to Hochschild homology, which is a theory developed by Hochschild some time before, in the years 1940. The link takes the form of an exact sequence, named *SBI* by Connes. Later, Beckmann worked on generalizing the notions of Hochschild and cyclic homologies to schemes over a field. The attempt is quite natural, as the category of algebras over a given field is equivalent (through a contravariant functor) to the category of affine schemes over this same field. We can thus wonder if the theory can be described for affine schemes in a way that could extend to schemes, maybe asking some good properties for the scheme. Beckmann did all that and once again linked it to K-theory, in his thesis [Beck]. In parallel, throughout the last centuries, the analytic number theory had been introduced and then studied. The work of Weil and Hasse on the fundation of the algebraic geometry, and the arithmetic geometry, with the study of *L-functions* in the p-adic case in the 30s' created a bridge between algebraic results and analytic number theory, because it allows to associate to any algebraic variety, or even more generally a motive over a number field an arithmetic function, that can be defined in any characteristic. Afterwards, with the intervention of Grothendieck and Serre, several conjectures for these *zeta functions* or L-functions have been made, as stated in the paper [Ser], using p-adic cohomology and Γ-functions. The study of these functions in positive characteristic is quite well known at the time being, but some analogies for the 0 characteristic are still to be made. Studying these functions in the years 1990, Deninger found a convenient way to express them by defining a new homology theory, which he called *archimedean*, as a call back to the *archimedean places* of a number field. In [Den], he therefore defines this theory, and gives a way to understand the L-function associated to a motive as the *regularized* determinant of a certain operator on this infinite dimensional space. The good way to regularize a determinant in order to achieve this is non trivial and detailed as well in his work. At the end of the 20th century, Connes began to work on the Riemann hypothesis, in collaboration with Marcolli, and then Consani. They tried to use a non-commutative point of view to refine the result of Deninger, defining a more natural archimedean homology which would carry once again the information about the L-function associated to a motive, which will be called as one could expect the *archimedean cyclic homology*. This homology is defined for any smooth projective variety over a number field (containing $\mathbb{Q}$), and is strongly related to the λ-decomposition of the cyclic homology, as treated by Loday in [Lod], or in the case of schemes by Weibel in [We2]. The aim of this mémoire is to explain and prove the theorem by Connes and Consani, as done in [CC], the theorem is stated as theorem 8.30

Theorem 1.1. *Let X be a smooth, projective variety of dimension d over an algebraic number field K, and let $\nu|\infty$ be an archimedean place of K ($K_\nu = \mathbb{R}$ or $\mathbb{C}$). Let $\Theta = \Theta_0 - \Gamma$ be the operator on $HC_*^{ar}(X_\nu)$, with Θ_0 the "generator" of the λ-operations on the homology (that is to say $\Theta_0 = j$ on $HC_*^{ar,(j)}(X_\nu)$), and Γ the grading operator (that is to say $\Gamma = n$ on $HC_n(X_\nu)$). Then the action* *of* Θ *satisfies the formula*

$$\prod_{0\leq m\leq 2d} L_\nu\left(H^m(X),s\right)^{(-1)^{m+1}} = \frac{\det_\infty\left(\frac{1}{2\pi}\left(s-\Theta\right)_{|HC^{ar}_{even}(X_\nu)}\right)}{\det_\infty\left(\frac{1}{2\pi}\left(s-\Theta\right)_{|HC^{ar}_{odd}(X_\nu)}\right)}, \tag{1.1.1}$$

for all complex numbers s. *The spaces labeled "even" and "odd" are defined as*

$$\begin{aligned} HC^{ar}_{even}(X_\nu) &= \bigoplus_{\substack{n\equiv 0[2],\\ n\geq 0}} HC^{ar}_n(X_\nu), \text{ and} \\ HC^{ar}_{odd}(X_\nu) &= \bigoplus_{\substack{n\equiv 1[2],\\ n\geq 1}} HC^{ar}_n(X_\nu). \end{aligned} \tag{1.1.2}$$

The spaces $HC^{ar,(j)}_*$ are the λ-decomposed archimedean cyclic homology groups, defined from the cyclic homology groups of the variety X_ν. First, in section 2, we will recall how to define Hochschild and cyclic homologies in the case of algebras, as well as the two variations which are the negative and periodic cyclic homologies, following what is done in [Lod], in the case of unital algebras. Then, in section 4.32, we will define the Kähler differential modules of an algebra in order to state the *Hochschild-Kostant-Rosenberg* theorem 3.11, which is a paramount tool to compute Hochschild and cyclic homology, and which reaches beyond the algebra case, and allows to find alternative definitions of the complexes used to define the cyclic homology of schemes. This will lead to the definition of the homology theories for schemes, in section 4, in which we will also define the λ-decomposition, which is a result emanating from basic bialgebra theory, but allows to split the homology as a direct sum of subspace, each of them being computable as a Deligne cohomology group, and that is the goal of section 5. After this section, we will aim towards the definition of the archimedean theories, beginning with the *Tate-twisted* cyclic homology, which takes into account the difference between the periodic homology of the manifold associated to a scheme when it is smooth over the real or complex numbers, and the periodic homology of the scheme itself. This gives rise to a *real cyclic homology*, which comes with a *Tate-twisted map* from it to the usual cyclic homology. When this is defined, we will be able to build the archimedean cyclic homology in section 7, at first in the complex case, and then in the real case, as the two slightly differ. Then we will be able to understand them as Deligne cohomology groups. When all of this is built, the theorem is easy to understand, explaining at first how to take the determinant of an operator over an infinite dimensional space, and this is done in section 8.30, leading to the statement and the proof of the theorem. Afterwards, we will treat the case of a fundamental example, that is the real and complex projective spaces of any dimension. The only example provided in [CC] was the case of schemes consisting of a single point, that are $\operatorname{Spec}\mathbb{R}$ and $\operatorname{Spec}\mathbb{C}$, which are included in the general computation that we give. Our computation relies on several proposition showed throughout the mémoire and used together to understand the cyclic homology of these schemes, and then their archimedean cyclic homology. In parallel, to check that the theorem makes sense in these cases, we will compute the L-functions associated to the schemes, by understanding their cohomology as *Hodge structures*. The examples are shown in section 9.

2. Preliminaries: Hochschild and cyclic homologies

2.1. **Hochschild homology.** We will first recall what are the Hochschild and cyclic homologies associated to an algebra A over a field k, with A not necessarily commutative. However, we will assume every algebra is unitary.

Definition 2.2. We define the *Hochschild complex* associated to the algebra A as

$$C_*(A) = \dots \xrightarrow{b} A^{\otimes n+1} \xrightarrow{b} A^{\otimes n} \xrightarrow{b} \dots \xrightarrow{b} A^{\otimes 3} \xrightarrow{b} A^{\otimes 2} \xrightarrow{b} A \to 0, \tag{2.2.1}$$

the tensor products being taken over k, with $C_0(A) = A$, and the operator b is defined as

(2.2.2)

$$b(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = (a_0 a_1) \otimes a_2 \otimes \cdots \otimes a_n + \sum_{i=1}^{n-1} (-1)^i a_0 \otimes a_1 \cdots \otimes (a_i a_{i+1}) \otimes \cdots \otimes a_n$$
$$+ (-1)^n (a_n a_0) \otimes a_1 \otimes \cdots \otimes a_{n-1}.$$

The operator b is such that $b \circ b = 0$, and $C_*(A)$ is therefore a complex. The *Hochschild homology* is defined as the homology of this complex.

$$HH_n(A) = H_n(C_*(A), b). \tag{2.2.3}$$

Definition 2.3. At the same time, we define the operator b' on the Hochschild complex as

$$b'(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = (a_0 a_1) \otimes a_2 \otimes \cdots \otimes a_n + \sum_{i=1}^{n-1} (-1)^i a_0 \otimes a_1 \cdots \otimes (a_i a_{i+1}) \otimes \cdots \otimes a_n. \tag{2.3.1}$$

Example 2.4. A classic example is the Hochschild homology of the $\mathbb{C}$-algebra. We have $HH_0(\mathbb{C}) = \mathbb{C}$, and all the other groups are trivial.

Remark 2.5. The computation can be done easily in that case, because the formula for b is directly usable. The result will be shown as a particular case of example 4.16, in which the computation is done for $\mathrm{Spec}(\mathbb{C}) \simeq \mathbb{P}^0_{\mathbb{C}}$ as a scheme.

2.6. **Cyclic homology.** To define the different cyclic homologies (usual, periodic and negative), we will define *Connes' bicomplex* $(\mathcal{B}_{**}(A), b, B)$, but in order to define this complex in the unital case, it can be useful to first define cyclic homology through the *Loday-Quillen* bicomplex, as in [Lod]. First, we define an action of $\mathbb{Z}/(n+1)\mathbb{Z}$ on $A^{\otimes n+1}$, called the *cyclic operator*.

Definition 2.7. For any n, we set

$$t(a_0 \otimes \cdots \otimes a_n) = (-1)^n (a_n \otimes a_0 \otimes \cdots \otimes a_{n-1}), \tag{2.7.1}$$

this operator is such that $t^{n+1} = 1$.

From this we define the *norm operator*.

Definition 2.8. We define the operator N on $A^{\otimes n+1}$ as

$$N = 1 + t + \cdots + t^n. \tag{2.8.1}$$

Proposition 2.9. *The operators defined above satisfy the relations*

$$\begin{aligned} (1-t)b' &= b(1-t), \\ b'N &= Nb. \end{aligned} \tag{2.9.1}$$

These operators are enough to define the *cyclic bicomplex*.

Definition 2.10. The cyclic bicomplex $B_{**}(A)$ as

(2.10.1)
$$\begin{tikzcd}
\arrow[d, "b"] & \arrow[d, "-b'"] & \arrow[d, "b"] & \arrow[d, "-b'"] & \\
A^{\otimes 3} \arrow[d, "b"] & A^{\otimes 3} \arrow[l, "1-t"'] \arrow[d, "-b'"] & A^{\otimes 3} \arrow[l, "N"'] \arrow[d, "b"] & A^{\otimes 3} \arrow[l, "1-t"'] \arrow[d, "-b'"] & \arrow[l, "N"'] \\
A^{\otimes 2} \arrow[d, "b"] & A^{\otimes 2} \arrow[l, "1-t"'] \arrow[d, "-b'"] & A^{\otimes 2} \arrow[l, "N"'] \arrow[d, "b"] & A^{\otimes 2} \arrow[l, "1-t"'] \arrow[d, "-b'"] & \arrow[l, "N"'] \\
A & A \arrow[l, "1-t"'] & A \arrow[l, "N"'] & A \arrow[l, "1-t"'] & \arrow[l, "N"']
\end{tikzcd},$$

the module in the left-hand corner is in position $(0,0)$, such that we have $B_{pq}(A) = C_q(A) = A^{\otimes q+1}$.

Definition 2.11. We define the *total complex* associated to this bicomplex $CC_*(A) = \mathrm{Tot}_* B_{**}(A)$, that is to say

(2.11.1)
$$CC_n(A) = \mathrm{Tot}_n(B_{**}(A)) = \bigoplus_{p+q=n} B_{pq}(A) = A^{\otimes n} \oplus A^{\otimes n-1} \oplus \cdots \oplus A.$$

the differential being the sum of the vertical and horizontal differentials of the bicomplex.

Definition 2.12. We define the cyclic homology of A to be the homology of this complex

(2.12.1)
$$HC_n(A) = H_n(CC_*(A)).$$

This complex can be rewritten, because the odd-indexed columns are acyclic, and therefore it is possible to make them disappear.

Proposition 2.13. *The complex*

(2.13.1)
$$\ldots \xrightarrow{-b'} A^{\otimes 3} \xrightarrow{-b'} A^{\otimes 2} \xrightarrow{-b'} A \to 0,$$

is null-homotopic, and thus it is also acyclic.

Proof. We build a map $s : A^{\otimes n} \to A^{\otimes n+1}$, called the *extra degeneracy* by setting

(2.13.2)
$$s(a_1 \otimes \cdots \otimes a_n) = (1 \otimes a_1 \otimes \cdots \otimes a_n).$$

A quick computation shows that it is such that

(2.13.3)
$$b's + sb' = 1,$$

that is to say s is a homotopy from the b'-complex 2.13.1 to the zero complex. □

Using a lemma proved in [Lod], called there the *killing contractible complexes* lemma, we obtain a quasi-isomorphism from the cyclic bicomplex to another bicomplex often called *Connes'* bicomplex.

Definition 2.14. We define a new map from the ones defined before:

(2.14.1)
$$B = (1-t)sN : A^{\otimes n} \to A^{\otimes n+1}.$$

Then the lemma allows us to delete the odd-indexed columns, which are repetitions of the b'-complex 2.13.1, filling the diagram with the new differential B as follows:

(2.14.2)
$$\begin{tikzcd}
{} \arrow[d, "b"] & {} \arrow[d, "b"] & {} \arrow[d, "b"] \\
A^{\otimes 3} \arrow[d, "b"] & A^{\otimes 3} \arrow[d, "b"] & A^{\otimes 3} \arrow[d, "b"] \\
A^{\otimes 2} \arrow[d, "b"] & A^{\otimes 2} \arrow[ul, "B"] \arrow[d, "b"] & A^{\otimes 2} \arrow[ul, "B"] \arrow[d, "b"] \\
A & A \arrow[ul, "B"] & A. \arrow[ul, "B"]
\end{tikzcd}$$

It is possible to change the indexation of this complex to get the usual (B, b)-bicomplex, denoted $\mathcal{B}_{**}(A)$,

(2.14.3)
$$\begin{tikzcd}
{} \arrow[d, "b"] & {} \arrow[d, "b"] & {} \arrow[d, "b"] \\
A^{\otimes 3} \arrow[d, "b"] & A^{\otimes 2} \arrow[l, "B"'] \arrow[d, "b"] & A \arrow[l, "B"'] \\
A^{\otimes 2} \arrow[d, "b"] & A \arrow[l, "B"'] & \\
A & &
\end{tikzcd} .$$

The lemma also insures that it is a bicomplex and thus that we have $Bb + bB = 0$. Explicitly, the map $B : A^{\otimes n+1} \to A^{\otimes n+2}$ is given by

(2.14.4)
$$B(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n} (-1)^{ni} \big(1 \otimes a_i \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{i-1}$$
$$- a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{i-1}\big) .$$

Theorem 2.15. *We get an injective map of complexes* $\mathrm{Tot}_*(\mathcal{B}_{**}(A)) \hookrightarrow CC_*(A)$*, which is a quasi-isomorphism, and therefore the cyclic homology can be computed as*

(2.15.1)
$$HC_n(A) = H_n(\mathrm{Tot}_*(\mathcal{B}_{**}(A))).$$

From this other way to compute the cyclic homology of an algebra, we get an interesting exact sequence relating Hochschild homology and cyclic homology.

Proposition 2.16. *There is a short exact sequence of bicomplexes*

(2.16.1)
$$0 \to C_*(A) \xrightarrow{I} \mathcal{B}_{**}(A) \xrightarrow{S} \mathcal{B}_{*-1,*-1}(A) \to 0,$$

seeing the complex $C_*(A)$ *concentrated in the first column. This sequence becomes, at the level of total complexes,*

(2.16.2)
$$0 \to C_*(A) \xrightarrow{I} CC_*(A) \xrightarrow{S} CC_{*-2}(A) \to 0,$$

the map S is really important and is called the periodicity map. *The long sequence of homology associated to this short sequence is called the* SBI sequence *and is of the form:*

$$\dots \xrightarrow{I} HC_n(A) \xrightarrow{S} HC_{n-2}(A) \xrightarrow{B} HH_{n-1}(A) \xrightarrow{I} HC_{n-1}(A) \xrightarrow{S} \dots . \tag{2.16.3}$$

At the end we have

$$\dots \xrightarrow{I} HC_1(A) \xrightarrow{S} 0 \xrightarrow{B} HH_0(A) \xrightarrow{I} HC_0(A) \xrightarrow{S} 0. \tag{2.16.4}$$

And thus we get $HC_0(A) \simeq HH_0(A) \simeq A/[A; A]$. We also get from this a spectral sequence

$$E^1_{p,q} = HH_{q-p}(A) \implies HC_{p+q}(A), \tag{2.16.5}$$

the differential d_1 being Connes' operator B.

Example 2.17. For the $\mathbb{C}$-algebra $\mathbb{C}$, we get (using the SBI sequence or again the example 4.16) $HC_{2n}(\mathbb{C}) \simeq \mathbb{C}$ and $HC_{2n+1}(\mathbb{C}) = 0$ for every natural number n.

2.18. **Periodic and negative cyclic homologies.** The are two other homology theories which can be useful, which are the periodic and negative cyclic homologies. They are obtained by completing the cyclic bicomplex $\mathcal{B}_{**}(A)$.

Definition 2.19. We define the periodic bicomplex T_{**} as the completion of the bicomplex $\mathcal{B}_{**}(A)$ in the left half-plane, that is to say $T_{pq} = A^{\otimes q}$ whenever $q \geq 0$, the other groups being trivial. The differentials are still alternatively N and $1-t$ for the horizontal one, and b and $-b'$ for the vertical ones. This bicomplex looks like

$$\begin{tikzcd}
 & {} \arrow[d, "b"] & {} \arrow[d, "-b'"] & {} \arrow[d, "b"] & {} \arrow[d, "-b'"] & {} \arrow[d, "b"] & \\
{} & A^{\otimes 3} \arrow[l, "N"'] \arrow[d, "b"] & A^{\otimes 3} \arrow[l, "1-t"'] \arrow[d, "-b'"] & A^{\otimes 3} \arrow[l, "N"'] \arrow[d, "b"] & A^{\otimes 3} \arrow[l, "1-t"'] \arrow[d, "-b'"] & A^{\otimes 3} \arrow[l, "N"'] \arrow[d, "b"] & {} \arrow[l, "1-t"'] \\
{} & A^{\otimes 2} \arrow[l, "N"'] \arrow[d, "b"] & A^{\otimes 2} \arrow[l, "1-t"'] \arrow[d, "-b'"] & A^{\otimes 2} \arrow[l, "N"'] \arrow[d, "b"] & A^{\otimes 2} \arrow[l, "1-t"'] \arrow[d, "-b'"] & A^{\otimes 2} \arrow[l, "N"'] \arrow[d, "b"] & {} \arrow[l, "1-t"'] \\
{} & A \arrow[l, "N"'] & A \arrow[l, "1-t"'] & {\color{red}A} \arrow[l, "N"'] & A \arrow[l, "1-t"'] & A \arrow[l, "N"'] & {} \arrow[l, "1-t"']
\end{tikzcd}, \tag{2.19.1}$$

the group colored in red being the one in position $(0, 0)$.

Definition 2.20. We can see the cyclic bicomplex as the bicomplex obtained by deleting the negatively numbered columns. Therefore, we define the negative cyclic bicomplex T^-_{**} as the bicomplex made of the columns of number lower than or equal to 1. This bicomplex comes with an obvious embedding I into T_{**}, which itself projects onto $\mathcal{B}_{**}(A)$.

Remark 2.21. The sequence of morphisms of complexes is not exact as the columns 0 and 1 are in the three complexes.

Definition 2.22. The periodic complex associated to the algebra A, $PC_*(A)$ is the *product total complex* of this bicomplex, that is to say

$$PC_n(A) = \mathrm{Tot}_n(T_{**}) = \prod_{p+q=n} T_{p,q} = \prod_{p=1}^{\infty} A^{\otimes p}, \tag{2.22.1}$$

with the differential being the sum of the horizontal and vertical differentials of T_{**}. The periodic homology of A is then defined as the homology of this complex

$$HP_n(A) = H_n(PC_*(A)). \tag{2.22.2}$$

Remark 2.23. It is crucial to take the *product* total complex and not the usual one, as the bicomplex T_{**} is not *biregular* (that is to say the diagonals are not finitely filled). The product total complex thus keeps a lot more information, and in the case of the periodic bicomplex, one can show that the homology of its sum total product is trivial. Moreover, if we consider a biregular bicomplex (e.g. Connes' bicomplex), then the two total complexes agree (because we are in an abelian category).

Proposition 2.24. *This periodic homology is really periodic as we have for any integer n*

$$PC_n(A) = \prod_{p=1}^{\infty} A^{\otimes p}, \tag{2.24.1}$$

with a 2-periodic differential,as it is easy to see on the bicomplex itself. Therefore for every integer n we have

$$HP_n(A) \simeq HP_{n+2}(A). \tag{2.24.2}$$

Definition 2.25. The negative complex associated to the algebra A, $NC_*(A)$ is defined as the product total complex of the bicomplex T^-_{**}, that is to say

$$NC_n(A) = \mathrm{Tot}_n(T^-_{**}) = \prod_{p+q=n} T^-_{p,q}. \tag{2.25.1}$$

For $n \leq 1$, we have $NC_n(A) = PC_n(A)$, but it is truncated if $n \geq 2$. The differential is as always defined as the sum of the vertical and horizontal ones. The negative cyclic homology of A is then defined as the homology of this complex

$$HN_n(A) = H_n(NC_*(A)). \tag{2.25.2}$$

Proposition 2.26. *There is a short exact sequence of complexes, induced by the identification of the translated cyclic bicomplex with the quotient of the periodic bicomplex by the negative bicomplex:*

$$0 \to NC_*(A) \to PC_*(A) \to CC_{*-2}(A) \to 0, \tag{2.26.1}$$

and thus a long exact sequence in homology

$$\cdots \to HN_{n+1}(A) \xrightarrow{I} HP_{n+1}(A) \xrightarrow{S} HC_{n-1}(A) \xrightarrow{B} HN_n(A) \to \dots. \tag{2.26.2}$$

Once again, it is possible, thanks to the *killing contractible complexes* lemma to rewrite this periodic bicomplex as the bicomplex obtained by completing Connes' (B, b)-bicomplex in the left half-plane.

Theorem 2.27. *The periodic bicomplex 2.19 is quasi-isomorphic to the following bicomplex, denoted $\mathcal{T}_{**}$,*

$$
\begin{tikzcd}
{} & {} \arrow[d, "b"] & {} \arrow[d, "b"] & {} \arrow[d, "b"] & {} \arrow[d, "b"] & {} \arrow[d, "b"] \\
{} & A^{\otimes 5} \arrow[l, "B"'] \arrow[d, "b"] & A^{\otimes 4} \arrow[l, "B"'] \arrow[d, "b"] & A^{\otimes 3} \arrow[l, "B"'] \arrow[d, "b"] & A^{\otimes 2} \arrow[l, "B"'] \arrow[d, "b"] & A \arrow[l, "B"'] \\
{} & A^{\otimes 4} \arrow[l, "B"'] \arrow[d, "b"] & A^{\otimes 3} \arrow[l, "B"'] \arrow[d, "b"] & A^{\otimes 2} \arrow[l, "B"'] \arrow[d, "b"] & A \arrow[l, "B"'] \\
{} & A^{\otimes 3} \arrow[l, "B"'] \arrow[d, "b"] & A^{\otimes 2} \arrow[l, "B"'] \arrow[d, "b"] & A \arrow[l, "B"'] \\
{} & A^{\otimes 2} \arrow[l, "B"'] \arrow[d, "b"] & {\color{red}A} \arrow[l, "B"'] \\
{} & A \arrow[l, "B"']
\end{tikzcd} \tag{2.27.1}
$$
,

the red A being located in position $(0,0)$. And therefore we have

$$HP_n(A) = H_n(\mathrm{Tot}_*(\mathcal{T}_{**})) \tag{2.27.2}$$

The same applies for the negative cyclic homology.

Theorem 2.28. *The negative bicomplex is quasi isomorphic to the subcomplex of $\mathcal{T}_{**}$ consisting of the columns such that $p \leq 0$, it is denoted $\mathcal{T}^{-}_{**}$. We have also*

$$HN_n(A) = H_n(\mathrm{Tot}_*(\mathcal{T}^{-}_{**})). \tag{2.28.1}$$

3. Differentials and Hochschild-Kostant-Rosenberg theorem

In the rest of this mémoire, many properties of cyclic homology come from an identification of certain algebras with algebras of Kähler differential forms on the algebra A considered. This correspondence comes from the *Hochschild-Kostant-Rosenberg* theorem. The definitions can be found on *nLab* or in the book [We3].

3.1. **Kähler differentials for an algebra.** To understand the idea of Kähler differentials, we will first define the first module with its universal property, as the "universal module equipped with a derivation". When we say that A is a commutative algebra, we mean a ring with a ring morphism $k \to A$, which naturally makes it an algebra over k.

Definition 3.2. Let k be a field and A a commutative k-algebra, and M a A-module. A *derivation* from A to M is a k-linear map $D : A \to M$ satisfying the product rule

$$D(ab) = D(a)b + aD(b), \tag{3.2.1}$$

for every elements a and b in A.

The first module of Kähler differentials is then defined with a universal property.

Proposition 3.3. *There exists a couple $(\Omega^1(A/k), d)$, where $\Omega^1(A/k)$ is a A-module and $d : A \to \Omega^1(A/k)$ is a derivation, such that for any A-module M with a derivation $D : A \to$, D factors uniquely through $\Omega^1(A/k)$, that is to say there exists a unique A-module morphism*

$$\mu : \Omega^1(A/k) \to M, \tag{3.3.1}$$

such that the following diagram commutes

$$\begin{tikzcd} A \arrow[rr, "D"] \arrow[dr, "d"'] & & M \\ & \Omega^1(A/k) \arrow[ur, "\mu"'] & \end{tikzcd} \quad . \tag{3.3.2}$$

Definition 3.4. The module of the previous proposition is called the first module of Kähler differentials over A.

There is a more direct construction of this module, as seen in the following proposition.

Proposition 3.5. *The A-module $\Omega^1(A/k)$ is isomorphic to the module generated by the symbols da, for every a in A, with the relations*

- *$dc = 0$ if c is an element of k seen as an element of A,*
- *$d(ab) = (da)b + a(db)$,*
- *$d(a + b) = da + db$,*
- *$(da)b = b(da)$,*

for a and b elements of A. The derivation d is obviously defined as $d(a) = da$.

From this description, we can see that the first Hochschild homology group of an algebra A always agrees with its module of Kähler differentials.

Theorem 3.6. *Let A be a commutative k-algebra, then $HH_0(A) \simeq A$, and there is an isomorphism*

$$HH_1(A) \simeq \Omega^1(A/k). \tag{3.6.1}$$

Proof. Recall the complex defining the Hochschild homology of an algebra A:

$$C_*(A) = \ldots \xrightarrow{b} A^{\otimes n+1} \xrightarrow{b} A^{\otimes n} \xrightarrow{b} \ldots \xrightarrow{b} A^{\otimes 3} \xrightarrow{b} A^{\otimes 2} \xrightarrow{b} A \to 0. \tag{3.6.2}$$

By definition, $HH_0(A) = A/[A;A]$, and if A is commutative, then $\mathrm{Im}(b_{|A\otimes A}) = [A;A] = 0$, and $HH_1(A) = (A \otimes_k A)/\mathrm{Im}(b_{|A^{\otimes 3}})$. For every a, b and c in A, we have

$$b(a \otimes b \otimes c) = ab \otimes c - a \otimes bc + ca \otimes b. \tag{3.6.3}$$

Writing this with the notation da, we have

$$b(adbdc) = abdc - ad(bc) + cadb. \tag{3.6.4}$$

That is to say that in $HH_1(A)$, we have the relations

$$ad(bc) = abdc + acdb, \tag{3.6.5}$$

which are equivalent to the relations given to define $\Omega^1(A/k)$. □

The goal of the Hochschild-Kostant-Rosenberg theorem is to relate the Kähler differential modules with Hochschild homology at higher degrees. For this we need to define the higher degree Kähler differential modules. These are often defined after the generalization to schemes, taking the sheafification of $\Omega^1(A/k)$ on the k-scheme $\operatorname{Spec} A$ and so on. However, we can define them right now in the case of algebras.

Definition 3.7. Suppose here that the field k has characteristic 0, then we define the higher degree Kähler differential modules as the exterior products of the first module. For a k-algebra A, we set $\Omega^0(A/k) = A$, and

$$\Omega^n(A/k) = \Lambda^n_A \Omega^1(A/k). \tag{3.7.1}$$

This definition allows us to define the *de Rham* complex associated with a commutative k-algebra.

Definition 3.8. We define the de Rham complex as the following

$$A = \Omega^0(A/k) \xrightarrow{d} \Omega^1(A/k) \xrightarrow{d} \Omega^2(A/k) \xrightarrow{d} \dots, \tag{3.8.1}$$

The differential d being defined as

$$d(a_0 da_1 \wedge da_2 \wedge \dots \wedge da_n) = 1da_0 \wedge da_1 \wedge \dots \wedge da_n. \tag{3.8.2}$$

The de Rham cohomology of the k-algebra A is then defined as the cohomology of this complex.

$$H^n_{dR}(A) = H^n(\Omega^*(A/k), d). \tag{3.8.3}$$

3.9. **Hochschild-Kostant-Rosenberg theorem and applications.** The preliminaries to the Hochschild-Kostant-Rosenberg theorem are the isomorphism for the degree 1 3.6, and the following, which gives a graded algebra morphism we would like to be an isomorphism.

Theorem 3.10. *Given a comuutative k-algebra, the isomorphism 3.6 extends to a differential graded ring morphism*

$$\psi : \Omega^*(A/k) \to HH_*(A). \tag{3.10.1}$$

Proof. To show that the first morphism extends to a morphism ψ as we want to get, it is enough to endow $HH_*(A)$ with a graded ring structure in order to use the universal property of the exterior product. In order to do so, we define the signed shuffle product:

$$\mu\left((a_1 \otimes \dots \otimes a_p), (a_{p+1} \otimes \dots \otimes a_{p+q})\right) = \sum_{\sigma\ (p,q)-\text{shuffle}} \varepsilon(\sigma)\sigma \cdot (a_1 \otimes \dots \otimes a_{p+q}). \tag{3.10.2}$$

With this product, checking the commutation with the operator B, the Hochschild complex is now endowed with a DG-algebra structure. Taking the homology, we get a graded commutative algebra structure on the Hochschild homology $HH_*(A)$. The existence of the morphism follows from this. □

Theorem 3.11. *Let k be a field and A a commutative k-algebra which satisfies the following properties:*

1. *A is finitely presented over k,*
2. *A is* smooth *over k, that is to say the A-module of Kähler differentials $\Omega^1(A/k)$ is a projective object in the category $A - \mathrm{Mod} - k$.*

Then the morphism

$$\psi : \Omega^*(A/k) \to HH_*(A), \tag{3.11.1}$$

is an isomorphism.

The proof rely on the regularity of the rings that appear and their localizations, and therefore uses dimension theory, and can be found in [We3].

Now we can see how this theorem is paramount in the computations of cyclic homology we are going to do in the rest of this mémoire.

Proposition 3.12. *With the hypotheses of theorem 3.11, it is trivial that the de Rham complex with the differential d replaced with 0 is quasi-isomorphic to the Hochschild complex and therefore computes the Hochschild homology too, the complex being*

$$A = \Omega^0(A/k) \xrightarrow{0} \Omega^1(A/k) \xrightarrow{0} \Omega^2(A/k) \xrightarrow{0} \dots. \tag{3.12.1}$$

From this we will want to replace the columns of Connes' (B, b)-bicomplex with copies of this de Rham complex with differential 0 instead of b. In order to do this, we must understand the behaviour of the map B under this correspondence.

Proposition 3.13. *For a commutative (non necessarily smooth) k-algebra A, with k a field of characteristic 0, the following diagram is commutative*

(3.13.1)
$$\begin{array}{ccc} \Omega^n(A/k) & \xrightarrow{d} & \Omega^{n+1}(A/k) \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\psi} \\ HH_n(A) & \xrightarrow{B_*} & HH_{n+1}(A). \end{array}$$

There is also another map $\pi : HH_*(A) \to \Omega^*(A/k)$, coming from the natural functorial map $C_*(A) \to \Omega^*(A/k)$, and we have $\pi_n \circ \psi_n = n! Id$.

Proposition 3.14. *Under the same hypotheses as the previous proposition, the following diagram is commutative*

(3.14.1)
$$\begin{array}{ccc} HH_n(A) & \xrightarrow{B_*} & HH_{n+1}(A) \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ \Omega^n(A/k) & \xrightarrow{(n+1)d} & \Omega^{n+1}(A/k). \end{array}$$

From this we see that when the algebra A satisfies the hypotheses of theorem 3.11, then we can replace the columns of Connes' bicomplex by shifted de Rham complexes, the vertical differentials being 0 and the horizontal being d.

Theorem 3.15. *Let A be a k-algebra which satisfies the conditions of theorem 3.11, with k of characteristic 0, then the bicomplex $B\Omega_{**}$ defined as*

(3.15.1)
$$\begin{array}{ccccccc} \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} \\ \Omega^3(A/k) & \xleftarrow[d]{} & \Omega^2(A/k) & \xleftarrow[d]{} & \Omega^1(A/k) & \xleftarrow[d]{} & A \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \\ \Omega^2(A/k) & \xleftarrow[d]{} & \Omega^1(A/k) & \xleftarrow[d]{} & A & & \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & & & \\ \Omega^1(A/k) & \xleftarrow[d]{} & A & & & & \\ \downarrow{\scriptstyle 0} & & & & & & \\ A & & & & & & \end{array}$$

is quasi-isomorphic to Connes' bicomplex $\mathcal{B}(A)$, and therefore we have

(3.15.2)
$$HC_n(A) = H_n(\mathrm{Tot}_*(B\Omega_{**})).$$

Moreover, this homology can be computed in terms of de Rham cohomology

(3.15.3)
$$HC_n(A) = \Omega^n(A/k)/d\Omega^{n-1}(A/k) \oplus H^{n-2}_{dR}(A) \oplus H^{n-4}_{dR}(A) \oplus \dots,$$

the last summand being $H^0_{dR}(A)$ or $H^1_{dR}(A)$ depending on the parity of n.

Remark 3.16. The same replacement is also possible for the periodic or negative cyclic homology.

4. Extension to schemes

In this section we will define what we would like to be a generalization of the Hochschild and cyclic homologies, extending the definition to schemes. For it to be a real generalization, it would require that the homology of any affine scheme agree with the homology of the corresponding algebra, that is to say, for any *affine* scheme X over a field k,

$$HH_n(X) = HH_n(\mathcal{O}_X(X)). \tag{4.0.1}$$

That is however not true in general but it becomes true with a few hypotheses on the scheme. From now on, when we talk about a k-algebra A, we mean a commutative unitary ring endowed with a ring morphism $k \to A$ which gives A a natural algebra structure. Moreover, when we talk about a k-scheme, or a scheme over the field k, we mean a scheme X endowed with a morphism of schemes $X \to \operatorname{Spec} k$. In these two categories, the morphisms are also supposed to make the obvious diagrams over k commute.

4.1. Hochschild homology of schemes and hypercohomology functors.

The definition of the Hochschild homology of a scheme is slightly different from the one for an algebra as the category on which we work is not the category of algebras over a give field, but of sheaves of algebras on the given scheme over the field. Therefore, in order to get a simple group and not a sheaf, we need to take the *hypercohomology* of the sheaf complex, which is a really convenient functor for the definitions, but not for the explicit computations, as it requires to find injective resolutions of the given complex. This definition seems to be first used by Beckmann in [Beck], but is then used by Weibel in [We1] and [We2].

Definition 4.2. Let $X = \operatorname{Spec}(A)$ be an affine scheme over a field k (A is a k-algebra), we define the Hochschild complex associated to the scheme as the sheafification of the Hochschild complex 2.2.1:

$$\dots \xrightarrow{b} A^{\otimes n+1} \xrightarrow{b} A^{\otimes n} \xrightarrow{b} \dots \xrightarrow{b} A \otimes_k A \otimes_k A \xrightarrow{b} A \otimes_k A \xrightarrow{b=0} A \to 0. \tag{4.2.1}$$

Generalizing this to the case where the scheme X is not affine, we set the Hochschild complex to be the following:

$$C_*(X) = \dots \xrightarrow{b} \mathcal{O}_X^{\otimes n+1} \xrightarrow{b} \mathcal{O}_X^{\otimes n} \xrightarrow{b} \dots \xrightarrow{b} \mathcal{O}_X \otimes_k \mathcal{O}_X \otimes_k \mathcal{O}_X \xrightarrow{b} \mathcal{O}_X \otimes_k \mathcal{O}_X \xrightarrow{b=0} \mathcal{O}_X \to 0. \tag{4.2.2}$$

We change this complex into a complex of cochains by setting $C^*(X) = C_{-*}(X)$, and switching the direction of the arrows. The *Hochschild homology* of the scheme X is then defined as the *hypercohomology* of this complex of sheaves, that is, for all integer n:

$$HH_n(X) = \mathbb{H}^{-n}(X, C^*(X)). \tag{4.2.3}$$

The hypercohomology functor is defined as follows:

Definition 4.3. Let $\mathcal{C}^*$ be a complex of sheaves on a scheme X, we take a *Cartan-Eilenberg resolution* of the complex, that is to say a bicomplex (I^{**}, d_1, d_2) with a family of monomorphisms $\mathcal{C}^k \to I^{k0}$, such that every complex

$$0 \to \mathcal{C}^k \xrightarrow{\varepsilon} I^{k0} \to I^{k1} \to \dots$$

is an injective resolution of $\mathcal{C}^k$.

$$
\begin{array}{ccccccccccc}
 & & \uparrow{\scriptstyle d_2} & & \uparrow{\scriptstyle d_2} & & \uparrow{\scriptstyle d_2} & & \uparrow{\scriptstyle d_2} & & \uparrow{\scriptstyle d_2} & \\
\xrightarrow{d_1} & I^{-2,2} & \xrightarrow{d_1} & I^{-1,2} & \xrightarrow{d_1} & I^{0,2} & \xrightarrow{d_1} & I^{1,2} & \xrightarrow{d_1} & I^{2,2} & \xrightarrow{d_1} \\
 & \uparrow{\scriptstyle d_2} & & \uparrow{\scriptstyle d_2} & & \uparrow{\scriptstyle d_2} & & \uparrow{\scriptstyle d_2} & & \uparrow{\scriptstyle d_2} & \\
\xrightarrow{d_1} & I^{-2,1} & \xrightarrow{d_1} & I^{-1,1} & \xrightarrow{d_1} & I^{0,1} & \xrightarrow{d_1} & I^{1,1} & \xrightarrow{d_1} & I^{2,1} & \xrightarrow{d_1} \\
 & \uparrow{\scriptstyle d_2} & & \uparrow{\scriptstyle d_2} & & \uparrow{\scriptstyle d_2} & & \uparrow{\scriptstyle d_2} & & \uparrow{\scriptstyle d_2} & \\
\xrightarrow{d_1} & I^{-2,0} & \xrightarrow{d_1} & I^{-1,0} & \xrightarrow{d_1} & I^{0,0} & \xrightarrow{d_1} & I^{1,0} & \xrightarrow{d_1} & I^{2,0} & \xrightarrow{d_1} \\
 & \uparrow{\scriptstyle \varepsilon} & & \uparrow{\scriptstyle \varepsilon} & & \uparrow{\scriptstyle \varepsilon} & & \uparrow{\scriptstyle \varepsilon} & & \uparrow{\scriptstyle \varepsilon} & \\
\xrightarrow{d} & \mathcal{C}^{-2} & \xrightarrow{d} & \mathcal{C}^{-1} & \xrightarrow{d} & \mathcal{C}^{0} & \xrightarrow{d} & \mathcal{C}^{1} & \xrightarrow{d} & \mathcal{C}^{2} & \xrightarrow{d}
\end{array} \tag{4.3.1}
$$

We can consider the *product total complex* of I^{**},

$$(\mathrm{Tot}^n I^{**}) = \prod_{k\in\mathbb{Z}} I^{k,n-k}, \tag{4.3.2}$$

with differential defined as $d = d_1 - d_2$. This complex is independent of the choice of the resolution I^{**} up to homotopy equivalence. The hypercohomology of $\mathcal{C}^*$ is then defined as the cohomology of the complex of global sections of this product total complex:

$$\mathbb{H}^i(X, \mathcal{C}^*) = H^i\left(\Gamma\left(\mathrm{Tot}^* I^{**}\right)\right) = H^i\left(\mathrm{Tot}^*\Gamma\left(I^{**}\right)\right). \tag{4.3.3}$$

Remark 4.4. We will sometimes omit the shift to cohomology when we work with homology theories. For instance, the notation

$$\mathbb{H}^n(X, \mathcal{C}_*(X)) \tag{4.4.1}$$

must in fact be understood as

$$\mathbb{H}^n(X, \mathcal{C}_*(X)) = \mathbb{H}^n(X, \mathcal{C}^*(X)), \tag{4.4.2}$$

where the raising of the star is the same as done in 4.2.

Remark 4.5. If the complex $\mathcal{C}^*$ is bounded below, then the product total complex of the Cartan-Eilenberg resolution is the same as the usual total complex, with direct sums instead of products as the families are finite. Therefore, the hypercohomology functor coincides with the usual right-derived functor associated to the global sections functor Γ, we have:

$$\mathbb{H}^i(X, \mathcal{C}^*) = \mathbb{R}^i\Gamma(\mathcal{C}^*). \tag{4.5.1}$$

Proposition 4.6. *This construction gives a Mayer-Vietoris sequence. Given a covering of X by two open sets U and V, we have an exact sequence for every complex $\mathcal{C}$:*

$$\cdots \to \mathbb{H}^i(X, \mathcal{C}) \to \mathbb{H}^i(U, \mathcal{C}_{|U}) \oplus \mathbb{H}^i(V, \mathcal{C}_{|V}) \to \mathbb{H}^i(U\cap V, \mathcal{C}_{|U\cap V}) \to \mathbb{H}^{i+1}(X, \mathcal{C}) \to \dots \tag{4.6.1}$$

Let us come back then to Hochschild homology. With this definition, there exists non trivial homology modules at negative integers. However, if X is quasi-separated and quasi-compact, only finitely many negative Hochschild modules can be non trivial.

Proposition 4.7. (Weibel, [We1]) *If X is a noetherian scheme over k of finite dimension d, and we have $HH_n(X) = 0$ for $n < -d$.*

In the case of an affine scheme, the Hochschild homology can be computed as the first homology sheaf of a quasi-coherent scheme. We define for every integer n the sheaf $\mathcal{HH}_n$ as the sheafification of the sheaf $U \mapsto HH_n(\mathcal{O}_X(U))$.

Proposition 4.8. (Weibel, [We1]) *For a scheme over a field k, the sheaves $\mathcal{HH}_n$ are quasi-coherent sheaves o X. Moreover, on an affine open $U = \operatorname{Spec} A$ of X, there are natural isomorphisms*

$$HH_n(A) \xrightarrow{\simeq} H^0(U, \mathcal{HH}_n).$$

Moreover the Hochschild homology of an affine scheme is, as one would expect, often equal to the usual Hochschild homology of the algebra.

Theorem 4.9. (Weibel, [We1]) *Let $X = \operatorname{Spec} A$ an affine scheme, with A a noetherian algebra of finite dimension, then we have isomorphisms for $n \geq 0$*

$$HH_n(X) \simeq HH_n(A).$$

4.10. **Cyclic homology of schemes.**

Definition 4.11. Similarly we define the cyclic homology of a scheme X over a field k by considering the double complex which is the sheafification of Connes' double complex $(\mathcal{B}_{**}(A), B, b)$ on $\operatorname{Spec} A$, defined in 2.14.3, which we denote again $\mathcal{B}_{**}(X)$. The cyclic homology of X is then defined as

$$HC_n(X) = \mathbb{H}^{-n}(X, \operatorname{Tot}_* \mathcal{B}_{**}(X)).$$

Proposition 4.12. *As in the case of algebras, the two homology theories are related by an SBI sequence, for any scheme X:*

$$\cdots \to HC_n(X) \xrightarrow{S} HC_{n-2}(X) \xrightarrow{B} HH_{n-1}(X) \xrightarrow{I} HC_{n-1}(X) \to \ldots. \tag{4.12.1}$$

Moreover, we get a spectral sequence

$$E^1_{p,q} = HH_{q-p}(X) \implies HC_{p+q}(X), \tag{4.12.2}$$

the differential d_1 being induced by the sheafification of the operator B of Connes' bicomplex.

As for the Hochschild homology, this definition agrees for affine schemes with good properties.

Theorem 4.13. (Weibel, [We1]) *Let $X = \operatorname{Spec} A$ an affine scheme, with A a noetherian algebra of finite dimension, then we have isomorphisms for $n \geq 0$*

$$HC_n(X) \simeq HC_n(A).$$

However, the definition of the cyclic and Hochschild homologies through hypercohomology is not really usable, unless there appears some way to get injective resolutions of the complexes (that can sometimes be done by replacing the complex by de Rham complex when that is possible using Hochschild-Kostant-Rosenberg theorem, as in section 3, see section 4.32). One way to compute it in easy cases is using links with the Čech cohomology of suitable sheaves as it is done in [Beck]. We will see what is necessary to compute cyclic homology of simple schemes such as projective spaces $\mathbb{P}^n_k$.

The main result is the following:

Proposition 4.14. *Let X be a noetherian separated scheme of finite dimension over the field k. Let $C^*(X)$ be the complex of sheaves for the Hochschild homology. Then there are isomorphisms for all integers n*

$$\check{H}^n(X, C^*(X)) \simeq \mathbb{H}^n(X, C^*(X)) \simeq HH_{-n}(X). \tag{4.14.1}$$

Where $\check{H}$ is the Čech cohomology functor for complexes of sheaves over a topological space.

4.15. **The example of the cyclic homology of the projective spaces over a field.**

Example 4.16. Let us compute the Hochschild and cyclic homology of the projective space of dimension 1 associated to a field k, $X = \mathbb{P}^1_k$. Recall that there is an affine covering of X given by the two open sets $U = (U_0, U_1)$, with

$$U_0 \simeq \operatorname{Spec}(k[T]),\ U_1 \simeq \operatorname{Spec}\left(k\left[\frac{1}{T}\right]\right),\ \text{and } U_0 \cap U_1 \simeq \operatorname{Spec}\left(k\left[T, \frac{1}{T}\right]\right).$$

The Hochschild homology is the same as the usual Hochschild homology. For any integer q, we have:

$$HH_q(U_0) \simeq HH_q(k[T]) \simeq \Omega^q(k[T]/k) \simeq \begin{cases} k[T] \text{ for } q = 0, \\ k[T]dT \text{ for } q = 1, \\ 0 \text{ for } q > 1, \end{cases} \tag{4.16.1}$$

and

$$HH_q(U_1) \simeq HH_q\left(k\left[\frac{1}{T}\right]\right) \simeq \Omega^q\left(k\left[\frac{1}{T}\right]/k\right) \simeq \begin{cases} k\left[\frac{1}{T}\right] \text{ for } q = 0, \\ k\left[\frac{1}{T}\right] d\frac{1}{T} \text{ for } q = 1, \\ 0 \text{ for } q > 1, \end{cases} \tag{4.16.2}$$

Moreover, on the intersection set we have:

$$HH_q(U_0 \cap U_1) \simeq \begin{cases} k\left[T, \frac{1}{T}\right] \text{ for } q = 0, \\ k\left[T, \frac{1}{T}\right] dT \text{ for } q = 1, \\ 0 \text{ for } q > 1, \end{cases} \tag{4.16.3}$$

From this, we can apply the hypercohomology spectral sequence to find the Hochschild homology of X:

$$E_2^{p,q} = \check{H}^p(U, H^q(C^*(X))) \implies HH_{-p-q}(X). \tag{4.16.4}$$

Moreover, we have isomorphisms:

$$\check{H}^p(U, H^q(C^*(X))) \simeq H^p\left(X, \Omega^{-q}_{X/k}\right) \simeq \begin{cases} k \text{ for } p = -q \text{ and } p \in \{0, 1\} \\ 0 \text{ otherwise,} \end{cases} \tag{4.16.5}$$

as in exercise 7.3 of chapter III of Hartshorne. The sheaf $\Omega^{-q}_{X/k}$ being the sheafification of the sheaf of Kähler differentials on the algebra A, $\Omega^{-q}(A/k)$ on $\operatorname{Spec} A$. Thus, the spectral sequence converges at page 2 and we get the following result:

$$HH_q(\mathbb{P}^1_k) \simeq \begin{cases} k^2 \text{ for } q = 0 \\ 0 \text{ otherwise,} \end{cases} \tag{4.16.6}$$

This method generalizes to $\mathbb{P}^n_k$ for any integer n.

$$HH_q(\mathbb{P}^n_k) \simeq \begin{cases} k^{n+1} \text{ for } q = 0 \\ 0 \text{ otherwise,} \end{cases} \tag{4.16.7}$$

Now we can deduce from the Hochschild homology the cyclic homology thanks to the spectral sequence 4.12.2. We have an upper-right quadrant spectral sequence

$$E^1_{p,q} = HH_{q-p}(X) \implies HC_{p+q}(X).$$

Knowing that $HH_{q-p}(X)$ is isomorphic to k^{n+1} if and only if $p = q$, the differential must be zero, and the sequence converges immediately, for all n. We get from this that $HC_m(X) \simeq HH_0(X) \simeq k^{n+1}$ if m is even, and 0 if m is odd. The SBI exact sequence 4.12.1 gives then

$$\cdots \to 0 \to HC_1(X) \to HC_{-1}(X) \to HH_0(X) \to HC_0(X) \to HC_{-2}(X) \to 0 \to \dots, \tag{4.16.8}$$

Weibel states in [We1] that the result can be obtained from this sequence, however, there is no easy way to see how $HC_{-1}(X)$ is zero but that is exactly what needs to be proved in order to get the first isomorphism. To avoid this issue, we will rely on another result to compute the cyclic homology of these schemes, as done by Keller in [Kel]. In this paper, Keller shows, using the existence of a *tilting object* in the derived category of sheaves of modules on the projective spaces, that the cyclic homology can be computed as the cyclic homology of a certain category, and that is easier to do. The result is

$$HC_*(\mathbb{P}^n_k) \xrightarrow{\simeq} HC_*(k)^{n+1}. \tag{4.16.9}$$

And the result follows from the cyclic homology of k as an algebra over itself. We have

$$HC_q(\mathbb{P}^n_k) \simeq \begin{cases} k^{n+1} \text{ for } q \geq 0,\ q \text{ even}, \\ 0 \text{ otherwise}, \end{cases} \tag{4.16.10}$$

4.17. **Periodic and negative cyclic homologies.** In this section, as we did working with algebras, we define the two other homology theories emerging from Connes' bicomplex, completing the complex in the upper-right and lower-left quadrants, the periodic and negative cyclic homology of schemes.

Definition 4.18. We define the periodic complex $\mathcal{T}_{**}$ as the sheafification of the periodic complex 2.19, with the same boundary operators as the complex $\mathcal{B}_{**}(X)$. This way, there is a canonical injection of the complex $\mathcal{B}_{**}(X) \hookrightarrow \mathcal{T}_{**}$, seeing it as the upper-right quadrant. We define the periodic cyclic homology of the scheme X as the hypercohomology of the product total complex of this complex, $HP_*(X)$. The subcomplex truncated, keeping only sheaves with negative first coordinate denoted NC_{**}, and the hypercohomology of its product total complex is called the negative cyclic homology of X and denoted $NC_*(X)$.

Proposition 4.19. *The three complexes fit in a short exact sequence of bicomplexes:*

$$0 \to NC_* \to PC_* \to CC_{*-2} \to 0.$$

The theory of hypercohomology gives therefore a long exact sequence

$$\cdots \to HN_n(X) \to HP_n(X) \to HC_{n-2}(X) \to HN_{n-1}(X) \to \dots \tag{4.19.1}$$

4.20. **The λ-decomposition for cyclic homology.** One of the most important tool in the following study is the λ-decomposition of the chain complexes, which is a decomposition of the sheaves composing the complexes as eigenspaces of an operator,the λ-operations. The construction extends to the homology and this decomposition is crucial for the result we seek to obtain. In fact, the operator comes from the general theory of bialgebras (called Hopf algebras in [Lod]).

4.20.1. *λ-operations in the general case.*

Definition 4.21. Let A be a k-vector space, where k is a field containing $\mathbb{Q}$, we define the *graded cotensor algebra* associated to A as

$$\mathcal{H} = T'(A) = \bigoplus_{n \geq 0} A^{\otimes n}, \tag{4.21.1}$$

where the tensor product are taken over k. The multiplication $\mu : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ is defined by the *signed shuffle*:

$$\mu\left((a_1 \otimes \cdots \otimes a_p), (a_{p+1} \otimes \cdots \otimes a_{p+q})\right) = \sum_{\sigma\ (p,q)-\text{shuffle}} \varepsilon(\sigma)\sigma \cdot (a_1 \otimes \cdots \otimes a_{p+q}). \tag{4.21.2}$$

The comultiplication $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ is given by the *deconcatenation*:

$$\Delta(a_1 \otimes \cdots \otimes a_p) = \sum_{i=0}^{p} \left((a_1 \otimes \cdots \otimes a_i), (a_{i+1} \otimes \cdots \otimes a_p)\right). \tag{4.21.3}$$

The unit is denoted $u : k \to \mathcal{H}$, being the identity on the first term, and the counit $c : \mathcal{H} \to k$, being the projector on the first term. With the multiplication and comultiplication, we can define the *convolution* of two k-linear maps $f, g : \mathcal{H} \to \mathcal{H}$, it is an associative law on $\mathrm{End}(\mathcal{H})$:

$$f * g = \mu \circ (f \otimes g) \circ \Delta. \tag{4.21.4}$$

This is enough to define the λ-operations in a general framework.

Definition 4.22. We define an endomorphism λ^j of $\mathcal{H}$, for every integer j by

$$\lambda^j = (Id_{\mathcal{H}})^{*j} = \mu^j \circ \Delta^j.$$

These endomorphisms are of degree zero, and we denote $\overline{\lambda}_n^j$ the restriction of λ^j to the subspace $\mathcal{H}_n = A^{\otimes n}$. At the same times, given a degree zero k-linear map $f : \mathcal{H} \to \mathcal{H}$ such that $f(1) = 0$, we define an endomorphism of $\mathcal{H}_n$ by setting

$$e^{(1)}(f) = \log(uc + f) = f - \frac{f^{*2}}{2} + \cdots + (-1)^{i+1}\frac{f^{*i}}{i} + \ldots, \tag{4.22.1}$$

they are well defined because by induction, f^{*i} is zero when restricted to $\mathcal{H}_n$ if $n < i$. We define from this endomorphism the following ones:

$$e^{(i)}(f) = \frac{\left(e^{(1)}(f)\right)^{*i}}{i!} \tag{4.22.2}$$

They are such that $e_n^{(i)} = 0$ if $n < i$.

Proposition 4.23. *For every integer p we have the formula:*

$$(uc + f)^{*p} = uc + \sum_{i \geq 1} p^i e^{(i)}(f). \tag{4.23.1}$$

Proof. Considering the formal series for exponential and logarithm,

$$\exp(X) = 1 + \sum_{i \geq 1} \frac{X^i}{i!}, \text{ and } \log(1 + X) = \sum_{i \geq 1} (-1)^{i+1} \frac{X^i}{i}, \tag{4.23.2}$$

they are related by the following general identity

$$(1 + X)^p = \exp(p \log(1 + X)). \tag{4.23.3}$$

We can apply this identity to f in the ring of endomorphisms of $\mathcal{H}$, endowed with the usual sum and the convolution, to get:

$$(uc + f)^{*p} = uc + \sum_{i \geq 1} p^i e^{(i)}(f). \tag{4.23.4}$$

$\square$

This formula applied to $f = Id - uc$ is what we are in fact interested in (we obviously have $f(1) = 0$).

Proposition 4.24. *We set for every integers i and n,*

$$e^{(i)} = e^{(i)}(Id - uc), \text{ and } e_n^{(i)} = e_n^{(i)}(Id - uc).$$

Then we have

$$(Id^{*p})_{|\mathcal{H}_n} = \sum_{i=1}^{n} p^i e_n^{(i)}. \tag{4.24.1}$$

Moreover for every integer n, the endomorphisms $e_n^{(i)}$ verify:

(1) $Id = e_n^{(1)} + \cdots + e_n^{(n)}$,
(2) $e_n^{(i)} e_n^{(j)} = \delta_{ij} e_n^{(i)}$

Proof. The relations 4.24.1 is just a rewriting of the previous identity. However, they represent n equations verified by the endomorphisms $e_n^{(i)}$, which matrix is a Vandermonde matrix M with $M_{p,i} = p^i$, which is invertible on $\mathbb{Q}$. Therefore, the $e_n^{(i)}$ are determined by the endomorphisms $(Id^{*p})_{|\mathcal{H}_n}$. But we have $Id^{*p} \circ Id^{*p'} = Id^{*pp'}$, and thus there is a unique formula of the form

$$e_n^{(i)} e_n^{(j)} = \sum_{m=1}^{n} a_{ijm} e_n^{(m)}.$$

Fixing m, we then get for all integers p and p'

$$\sum_{1 \le i,j \le n} p^i p'^j a_{ijm} = (pp')^m. \tag{4.24.2}$$

The only solution to this equation is given by the second point of the proposition. $\square$

4.24.1. *λ-decomposition of Hochschild and cyclic homologies.* We have defined general operations on the cotensor Hopf algebra associated to a k-algebra. These operators behave well with the boundaries of the Hochschild complex and allow us to split it into subcomplexes that will give the *λ-decomposition* for Hochschild homology.

Proposition 4.25. *Let A be a commutative k-algebra, and $C_*(A) = A \otimes_k \mathcal{H}$ its Hochschild complex as in 2.2.1, where $\mathcal{H}$ is its cotensor Hopf algebra, with the extension of the λ-operations by $id_A \otimes \overline{\lambda}_n^p$. The Hochschild operator b acts on $C_*(A)$, it is an operator of degree -1. For every integers $n \ge 1$ and $p \ge 0$, we have:*

$$b\overline{\lambda}_n^p = \overline{\lambda}_{n-1}^p b, \tag{4.25.1}$$

and

$$be_n^{(p)} = e_{n-1}^{(p)} b \tag{4.25.2}$$

Proof. As $\overline{\lambda}^p = \mu^p \circ \Delta^j$ on $\mathcal{H}$, is it enough to show that the map b on $C_*(A)$ and $b \otimes 1_{\mathcal{H}} + 1_{\mathcal{H}} \otimes b$ commute with $1_A \otimes \Delta$ and $1_A \otimes \mu$. That can be verified easily for Δ. For μ, it is a consequence of the fact that the boundary b is a graded derivation for the shuffle product. The same applies for the second equality. $\square$

Theorem 4.26. *Let A be a commutative k-algebra, where k is a field that contains $\mathbb{Q}$, then the idempotents $e_n^{(i)}$ split the Hochschild complex $C_*(A)$ into a sum of sub-complexes $C_*^{(i)}(A)$ for $i \ge 0$, whose homology $HH_*^{(i)}(A)$ satisfies $HH_0(A) = HH_0^{(0)}(A)$ and for all $n \ge 1$*

$$HH_n(A) = \bigoplus_{i=1}^{n} HH_n^{(i)}(A). \tag{4.26.1}$$

Proof. We set for all integers i and n, $C_n^{(i)}(A) = e_n^{(i)} C_n(A)$, the image of $C_n(A)$ under the projector $e_n^{(i)}$. As A is commutative, b is zero on $C_1(A)$. As $e_0^{(0)}$ is equal to $Id_{|C_0}$, the formula is trivially true for $n = 0$. For $n \geq 1$, the formula 4.24 gives the decomposition

$$C_n(A) = C_n^{(1)}(A) \oplus \cdots \oplus C_n^{(n)}(A).$$

Then proposition 4.25 shows that each $C_*^{(i)}(A)$ is a subcomplex, and more presicely a direct summand of $C_*(A)$. Taking the homology gives the decomposition that we wanted. $\square$

Theorem 4.27. (Loday) *With the same hypothesis as before, we have an isomorphism*

$$\Omega^n(A/k) \simeq HH_n^{(n)}(A). \tag{4.27.1}$$

Remark 4.28. Note that the previous theorem shows that if the algebra A is *smooth* in the sense of theorem 3.11, then we have

$$HH_n(A) \simeq HH_n^{(n)}(A) \simeq \Omega^n(A/k), \tag{4.28.1}$$

and therefore

$$HH_n^{(j)}(A) = 0 \tag{4.28.2}$$

for all integers n and $j \neq n$.

To extend this decomposition to cyclic homology, it is only necessary to see that the λ-operations behave well with Connes' boundary operator B on the cylic complex.

Theorem 4.29. *We have for integers i and n:*

$$Be_n^{(i)} = e_{n+1}^{(i+1)} B. \tag{4.29.1}$$

For every integer n, the cyclic homology splits in a sum as follows, $HC_0(A) = HC_0^{(0)}(A)$ and

$$HC_n(A) = \bigoplus_{i=1}^{n} HC_n^{(i)}(A). \tag{4.29.2}$$

The same holds from the exact same arguments for periodic and negative cyclic homologies.

Theorem 4.30. (Loday) *If the algebra A is smooth in the sense of theorem 3.11, the λ-decomposition agrees with the decomposition in terms of de Rham cohomology, as obtained in the computation 3.15.3. More precisely, we have*

$$HC_n^{(n)}(A) = \Omega^n(A/k)/d\Omega^{n-1}(A/k), \tag{4.30.1}$$

and

$$HC_n^{(j)}(A) = H_{dR}^{2j-n}(A), \quad \text{if } \left\lfloor \frac{n}{2} \right\rfloor \leq j < n, \tag{4.30.2}$$

and finally

$$HC_n^{(j)}(A) = 0, \quad \text{if } \left\lfloor \frac{n}{2} \right\rfloor \geq j. \tag{4.30.3}$$

Remark 4.31. The definition of the λ-operations extends to the case of a scheme X over a field k which contains $\mathbb{Q}$, and the splitting into a direct sum of spaces indexed by integers as in the theorems before stays true. However, it is necessary to pay attention to the fact the integers j may not have to be between 0 and n to have a nontrivial group $HC_n^{(j)}(X)$, as for example $HC_{-1}(X)$ can be nontrivial.

4.32. **Correspondence of the λ-decomposition with cohomology of suitable sheaves.** First of all, when $i = 0$, we can express easily the groups $HH_n^{(0)}(X)$.

Proposition 4.33. *If X is a scheme over $\mathbb{C}$, then when $i = 0$, we have $C_*^{(0)}(X) = \mathcal{O}_X$, and therefore its hypercohomology is simply cohomology and we get*

$$HH_n^{(0)}(X) = \begin{cases} H^{-n}(X, \mathcal{O}_X) \text{ if } n \leq 0, \\ 0 \text{ otherwise.} \end{cases} \tag{4.33.1}$$

Proof. Let (I^*, d) be an injective resolution of the sheaf $\mathcal{O}_X$. Then a Cartan-Eilenberg resolution of the part $i = 0$ of the Hochschild complex can be taken as

$$\begin{tikzcd}
 & {} & {} & {} & {} & {} & \\
{} \arrow[r, "0"] & 0 \arrow[u, "0"] \arrow[r, "0"] & 0 \arrow[u, "0"] \arrow[r, "0"] & I^3 \arrow[u, "d"] \arrow[r, "0"] & 0 \arrow[u, "0"] \arrow[r, "0"] & 0 \arrow[u, "0"] \arrow[r, "0"] & {} \\
{} \arrow[r, "0"] & 0 \arrow[u, "0"] \arrow[r, "0"] & 0 \arrow[u, "0"] \arrow[r, "0"] & I^2 \arrow[u, "d"] \arrow[r, "0"] & 0 \arrow[u, "0"] \arrow[r, "0"] & 0 \arrow[u, "0"] \arrow[r, "0"] & {} \\
{} \arrow[r, "0"] & 0 \arrow[u, "0"] \arrow[r, "0"] & 0 \arrow[u, "0"] \arrow[r, "0"] & I^1 \arrow[u, "d"] \arrow[r, "0"] & 0 \arrow[u, "0"] \arrow[r, "0"] & 0 \arrow[u, "0"] \arrow[r, "0"] & {} \\
{} \arrow[r, "0"] & 0 \arrow[u, "0"] \arrow[r, "0"] & 0 \arrow[u, "0"] \arrow[r, "0"] & I^0 \arrow[u, "d"] \arrow[r, "0"] & 0 \arrow[u, "0"] \arrow[r, "0"] & 0 \arrow[u, "0"] \arrow[r, "0"] & {} \\
{} \arrow[r, "0"] & 0 \arrow[u, "0"] \arrow[r, "0"] & 0 \arrow[u, "0"] \arrow[r, "0"] & \mathcal{O}_X \arrow[u, "\varepsilon"] \arrow[r, "0"] & 0 \arrow[u, "0"] \arrow[r, "0"] & 0 \arrow[u, "0"] \arrow[r, "0"] & {} .
\end{tikzcd} \tag{4.33.2}$$

The total complex of the Cartan-Eilenberg resolution is therefore just the resolution I^*, and therefore its cohomology is (up to reindexation) by definition the usual sheaf cohomology of $\mathcal{O}_X$. □

Proposition 4.34. *Let $X_{\mathbb{C}}$ be the scheme over $\mathbb{C}$ associated to a smooth projective complex algebraic variety X_{alg} (the points are different but the categories of open sets agree). For all integers n and j, we can compute the Hochschild homology as*

$$HH_n^{(j)}(X_{\mathbb{C}}) \simeq H^{j-n}\left(X_{\mathbb{C}}, \Omega^j_{X_{\mathbb{C}}/\mathbb{C}}\right) \tag{4.34.1}$$

the sheaves $\Omega^j_{X_{\mathbb{C}}/\mathbb{C}}$ being once again the sheafification of the modules of Kähler differentials of order j on $X_{\mathbb{C}}$.

The Hochschild-Kostant-Rosenberg theorem, 3.11, links the Hochschild homology complex to the algebra of algebraic differential forms associated to the algebra as it allows to replace in the computation of the Hochschid homology of a scheme X, the complex of sheaves by the complex of sheaves $\Omega^*_{X_{\mathbb{C}}/\mathbb{C}}$ of algebraic differential forms of degree $*$, with the boundary replace by the zero morphism, just as we did in section 3 in the case of algebras. The new bicomplex is the following.

(4.34.2)
$$\begin{tikzcd}
\arrow[d, "0"] & \arrow[d, "0"] & \arrow[d, "0"] & \arrow[d, "0"] \\
\Omega^3_{X_{\mathbb{C}}/\mathbb{C}} \arrow[d, "0"] & \Omega^2_{X_{\mathbb{C}}/\mathbb{C}} \arrow[l, "d"'] \arrow[d, "0"] & \Omega^1_{X_{\mathbb{C}}/\mathbb{C}} \arrow[l, "d"'] \arrow[d, "0"] & \mathcal{O}_{X_{\mathbb{C}}} \arrow[l, "d"'] \\
\Omega^2_{X_{\mathbb{C}}/\mathbb{C}} \arrow[d, "0"] & \Omega^1_{X_{\mathbb{C}}/\mathbb{C}} \arrow[l, "d"'] \arrow[d, "0"] & \mathcal{O}_{X_{\mathbb{C}}} \arrow[l, "d"'] & \\
\Omega^1_{X_{\mathbb{C}}/\mathbb{C}} \arrow[d, "0"] & \mathcal{O}_{X_{\mathbb{C}}} \arrow[l, "d"'] & & \\
\mathcal{O}_{X_{\mathbb{C}}} & & &
\end{tikzcd}$$

The Cartan-Eilenberg resolution can then be taken with the horizontal boundary being identically zero. The λ-decomposition is then read as the decomposition of the resolution bicomplex as sum of its columns.

The same applies for cyclic homology, we can replace Connes' bicomplex by the following $(\Omega^*_{X_{\mathbb{C}}/\mathbb{C}}, 0, d)$, where d is the usual de Rham differential. The total complex of this bicomplex becomes

$$\mathcal{T}_n = \bigoplus_{m\geq 0} \Omega^{n-2m}_{X_{\mathbb{C}}/\mathbb{C}}, \text{ with boundary operator } d\left(\sum_{m\geq 0} \omega_{n-2m}\right) = \sum_{m\geq 1} d\omega_{n-2m} \in \mathcal{T}_{n-1}.$$

Passing to the corresponding complex of cochains $\mathcal{T}^* = \mathcal{T}_{-*}$, we get the following decomposition

$$\mathcal{T}^n = \bigoplus_{0\leq j\leq -n} \Omega^{2j+n}_{X_{\mathbb{C}}/\mathbb{C}}, \tag{4.34.3}$$

and so

$$(\mathcal{T}^*, d) = \bigoplus_{j\geq 0} \left(\Omega^{\leq j}_{X_{\mathbb{C}}/\mathbb{C}}, d\right)[-2j], \tag{4.34.4}$$

corresponding to the λ-decomposition of the cyclic homology complex. Selecting the corresponding parts of the Cartan-Eilenberg reslution, we get the following.

Proposition 4.35. *For all integers n and j, we have*

$$HC^{(j)}_n(X_{\mathbb{C}}) = \mathbb{H}^{2j-n}\left(X_{\mathbb{C}}, \Omega^{\leq j}_{X_{\mathbb{C}}/\mathbb{C}}\right), \tag{4.35.1}$$

and thus,

$$HC_n(X_{\mathbb{C}}) = \prod_{j\in\mathbb{Z}} \mathbb{H}^{2j-n}\left(X_{\mathbb{C}}, \Omega^{\leq j}_{X_{\mathbb{C}}/\mathbb{C}}\right). \tag{4.35.2}$$

5. Deligne cohomology and correspondence with the λ-decomposition of cyclic homology

In this section we will see how spaces obtained in the previous section thanks to the λ-operations can be computed through another homology theory in certain cases, that is the (potentially reduced) Deligne cohomology of a scheme over $\mathbb{R}$ or $\mathbb{C}$. In most of the cases, we need to suppose X to be a smooth projective variety over one of those fields.

5.1. **The reduced Deligne cohomology.** In this section, let X be a smooth projective scheme over the complex numbers.

Definition 5.2. We denote for every integer r by $\mathbb{R}(r)$ the additive subgroup $(2\pi i)^r\mathbb{R}$ of $\mathbb{C}$, and by $\mathbb{R}(r)_{\mathcal{D}}$ the truncated complex of sheaves of holomorphic differential forms of type $(p,0)$, for p between 0 and $r-1$, on the complex manifold $X(\mathbb{C})$ associated to the complex points of X with $\mathbb{R}(r)$ placed in degree 0:

$$\mathbb{R}(r)_{\mathcal{D}} = 0 \to \mathbb{R}(r) \xrightarrow{\varepsilon} \Omega^0_{X(\mathbb{C})} \xrightarrow{\partial} \Omega^1_{X(\mathbb{C})} \xrightarrow{\partial} \dots \xrightarrow{\partial} \Omega^{r-1}_{X(\mathbb{C})} \to 0, \tag{5.2.1}$$

the map ε is the inclusion of elements of $\mathbb{R}(r)$ as constant holomorphic functions on $X(\mathbb{C})$. The real Deligne cohomology of X is defined as the hypercohomology of this complex:

$$H^n_{\mathcal{D}}(X_{\mathbb{C}}, \mathbb{R}(r)) = \mathbb{H}^n(X(\mathbb{C}), \mathbb{R}(r)_{\mathcal{D}}). \tag{5.2.2}$$

Definition 5.3. There is a short exact sequence of complexes

$$0 \to \left(\Omega^{<r}_{X(\mathbb{C})}\right)[1] \to \mathbb{R}(r)_{\mathcal{D}} \to \mathbb{R}(r) \to 0. \tag{5.3.1}$$

We call the first complex of the sequence the *reduced Deligne complex*, and we define the *reduced Deligne cohomology* to be the hypercohomology of this complex:

$$\tilde{H}^n_{\mathcal{D}}(X_{\mathbb{C}}, \mathbb{R}(r)) = \mathbb{H}^n\left(X(\mathbb{C}), \left(\Omega^{<r}_{X(\mathbb{C})}\right)[1]\right) = \mathbb{H}^n\left(X(\mathbb{C}), \mathrm{Cone}(F^r\Omega^*_{X(\mathbb{C})} \xrightarrow{\iota} \Omega^*_{X(\mathbb{C})})[1]\right), \tag{5.3.2}$$

where $F^r\Omega^*_{X(\mathbb{C})}$ is the subcomplex of $\Omega^*_{X(\mathbb{C})}$ of differential forms of degree greater than or equal to r. As an analogy, we will write

$$H^n_{dR}(X(\mathbb{C}))/F^r = \tilde{H}^{n+1}_{\mathcal{D}}(X_{\mathbb{C}}, \mathbb{R}(r)) = \mathbb{H}^n\left(X(\mathbb{C}), \left(\Omega^{<r}_{X(\mathbb{C})}\right)\right), \tag{5.3.3}$$

for all integers n and r.

If we consider a smooth projective variety X over a number field K, then we must precise what we mean by *Deligne cohomology* of X_ν when $\nu|\infty$ is an archimedean place depending on the fact that ν can be either real or complex.

Definition 5.4. For a variety X as above, if the place ν is complex, then we define the real Deligne cohomology groups as in 5.2.2, with $X_\nu = X_{\mathbb{C}}$. However, if the place is real, then they are defined as

$$H^q_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(r)) = H^q_{\mathcal{D}}(X_{\mathbb{C}}, \mathbb{R}(r))^{F=id}, \tag{5.4.1}$$

where $X_{\mathbb{C}}$ is defined as the complexification of the variety $X_{\mathbb{R}}$, i.e.

$$X_{\mathbb{C}} = X_{\mathbb{R}} \times_{\mathrm{Spec}\,\mathbb{R}} \mathrm{Spec}\,\mathbb{C}. \tag{5.4.2}$$

That is to say the elements of the previously defined Deligne cohomology of the complex variety associated to $X_{\mathbb{R}}$, fixed by the de Rham conjugation denoted F, defined as the conjugation of the coefficient of the function taken for the form. The fixed points correspond to the forms with real coefficients.

5.5. **Relation between cyclic homology and reduced Deligne cohomology.**

Theorem 5.6. *Let X be a smooth projective algebraic variety over $\mathbb{C}$. Then there are canonical isomorphisms, for all non negative integers n and j:*

$$HC^{(j)}_n(X) \simeq H^{2j-n}_{dR}(X(\mathbb{C}), \mathbb{C})/F^{j+1} = \tilde{H}^{2j+1-n}_{\mathcal{D}}(X, \mathbb{R}(j+1)). \tag{5.6.1}$$

Proof. The identification is a result of the comparison of the algebraic de Rham cohomology on $X_{\mathbb{C}}$ and the analytic de Rham cohomology on $X(\mathbb{C})$. □

Remark 5.7. The de Rham cohomology, whether algebraic or analytical, is naturally isomorphic to the topological cohomology of $X(\mathbb{C})$, which we will call the *Betti cohomology*, and denote H^n_B. The filtration F^* is the filtration by the degree of the differential forms.

The goal of this mémoire is to obtain an expression of an algebraic L-function associated to a variety in terms of a determinant on an infinite dimensional space which will be an *archimedean* version of the cyclic homology. The previous isomorphism raises the following equality.

Corollary 5.8. *Let X be a variety as above, of dimension d, then we have an isomorphism*

$$\bigoplus_{n\geq 0} HC_n(X_{\mathbb{C}}) \simeq \bigoplus_{(n,j)\in E_d} \tilde{H}^{2j+1-n}_{\mathcal{D}}(X_{\mathbb{C}}, \mathbb{R}(j+1)), \tag{5.8.1}$$

the set E_d being defined as a subset of $\mathbb{Z}^2$ as follows

$$E_d = \left\{(n,j)\in\mathbb{Z}^2 \,|\, 0\leq n\leq 2j\leq 2d+n\right\}. \tag{5.8.2}$$

Proof. The sum is at first taken over any pair of integers, but the previous theorem 5.6 insures that the group $HC^{(j)}_n(X)$ must be zero when $2j-n$ is not between 0 and the real dimension $2d$ of the variety. □

In parallel to this identification, Beilinson proved that the Deligne cohomology carries the information about the poles of the L-functions associated to a variety or more generally a motive (as seen in [Den]). The L-functions will be introduced in section 8, defined as a shifted product of Γ-functions, but the formula can be stated right now, in order to understand the difference with the cyclic homology as we defined it.

Theorem 5.9. *Let M be a motive over a number field k (or in our case, a smooth projective variety of dimension d). Let also w be a* Hodge weight *for M, and ν be an infinite (archimedean) place for k. Then for every integer m such that $m\leq w/2$, we have*

$$\mathrm{ord}_{s=m} L_\nu \left(H^w(M), s\right)^{-1} = \dim_{\mathbb{R}} H^{w+1}_{\mathcal{D}}(M_\nu, \mathbb{R}(w+1-m)). \tag{5.9.1}$$

The Deligne cohomology is different if the place ν is real or complex, as in definition 5.4. Moreover, this formula shows every possible pole for the considered L-function.

Definition 5.10. Therefore, this theorem leads us to consider the following infinite dimensional space in order to express the L-function as a regularized determinant (see again section 8), for an algebraic variety of a number field k, of dimension d,

$$\mathcal{H}^{ar} = \bigoplus_{(m,w)\in A_d} H^{w+1}_{\mathcal{D}}(X_\nu, \mathbb{R}(w+1-m)), \tag{5.10.1}$$

where the subset A_d of $\mathbb{Z}^2$ is defined as

$$A_d = \left\{(m,w)\in\mathbb{Z}^2 \,|\, 0\leq w\leq 2d,\, m\leq w/2\right\}. \tag{5.10.2}$$

However, there is a bijection between A_d and E_d which allows to bring out the similarities of this space and the direct sum of the cyclic homology groups of the variety X.

Proposition 5.11. *We define a bijection by setting*

$$\begin{aligned} f : E_d &\to A_d \\ (n,j) &\mapsto (j-n, 2j-n), \end{aligned} \tag{5.11.1}$$

its inverse being given by

$$f^{-1} : A_d \to E_d \\ (m, w) \mapsto (w - 2m, w - m). \tag{5.11.2}$$

Under this bijection we can rewrite $\mathcal{H}^{ar}$ *as*

$$\mathcal{H}^{ar} = \bigoplus_{(m,w)\in A_d} H_{\mathcal{D}}^{w+1}(X_\nu, \mathbb{R}(w+1-m)) = \bigoplus_{(n,j)\in E_d} H_{\mathcal{D}}^{2j+1-n}(X_\nu, \mathbb{R}(j+1)). \tag{5.11.3}$$

Which is therefore the same sum as in 5.8, taking the usual Deligne cohomology instead of the reduced one.

The bijection between the sets A_d and E_d is in fact paramount for the proofs we are going to detail further in this document, as it allows to go back and forth between Deligne cohomology and what we are going to define as the *archimedean cyclic homology*.

5.12. **Relation between reduced and classic Deligne homology.** Let X be a variety as above. We first assume that the place ν is complex. In that case, the short sequence 5.3 gives a long exact sequence

(5.12.1)

$$\cdots \to H_B^w(X(\mathbb{C}), \mathbb{R}(r)) \to \tilde{H}_{\mathcal{D}}^{w+1}(X_{\mathbb{C}}, \mathbb{R}(r)) \to H_{\mathcal{D}}^{w+1}(X_{\mathbb{C}}, \mathbb{R}(r)) \to H_B^{w+1}(X(\mathbb{C}), \mathbb{R}(r)) \to \dots,$$

The terms which interest us in these sequences are the groups $H_{\mathcal{D}}^{w+1}(X_{\mathbb{C}}, \mathbb{R}(r))$ for $(w, m) \in A_d$, with $r = w + 1 - n$.

Proposition 5.13. *Whenever* $w + 1 < 2r$*, there is a short exact sequence*

$$0 \to H_B^w(X(\mathbb{C}), \mathbb{R}(r)) \to \tilde{H}_{\mathcal{D}}^{w+1}(X_{\mathbb{C}}, \mathbb{R}(r)) \to H_{\mathcal{D}}^{w+1}(X_{\mathbb{C}}, \mathbb{R}(r)) \to 0. \tag{5.13.1}$$

Proof. First, for $w < 2r$, the natural map

$$H_B^w(X(\mathbb{C}), \mathbb{R}(r)) \to H_{dR}^w(X(\mathbb{C}), \mathbb{C})/F^r \tag{5.13.2}$$

is injective (see [Hu]). Therefore we can split the long exact sequence in shorter ones, for $(w, w + 1 - r) \in A_d$, because in that case, we have

$$w + 1 - r < w/2, \text{ and thus } w/2 + 1 < r, \text{ and finally } w + 1 < 2r, \tag{5.13.3}$$

which is enough to split the sequence. □

Remark 5.14. Everything works here in the case where ν is a complex place of the field k, however, this does not apply to the real case, for which the study must be refined.

6. Real Tate-twisted cyclic homology

In this section we will study the differences between the homologies of the schemes $X_{\mathbb{C}}$ and $\mathrm{Spec}(\mathcal{C}^\infty(X_{sm}, \mathbb{C}))$, when $X_{\mathbb{C}}$ is a smooth projective complex variety, and X_{sm} is the smooth complex manifold underlying $X_{\mathbb{C}}(\mathbb{C})$, and from this define the *real cyclic homology*, together with a *Tate-twisted* map. However, to correctly define these, it is necessary to take into account the Frechet topology of $\mathrm{Spec}(\mathcal{C}^\infty(X_{sm}, \mathbb{C}))$ in the definition of the periodic homology of the algebra.

6.1. **From $X_{\mathbb{C}}$ to** $\mathrm{Spec}(\mathcal{C}^\infty(X_{sm},\mathbb{C}))$**.** Given $X_{\mathbb{C}}$ a smooth projective complex variety viewed as a scheme, and X_{alg} its set of closed points. There is a canonical morphism of locally ringed spaces

$$\mu : X_{sm} \to X_{alg}, \tag{6.1.1}$$

endowing X_{sm} with the structure sheaf $\mathcal{O}_{X_{sm}}(U) = \mathcal{C}^\infty(U,\mathbb{C})$. This morphism can actually be extended to a morphism of schemes

$$\pi_X : \mathrm{Spec}(\mathcal{C}^\infty(X_{sm},\mathbb{C})) \to X_{\mathbb{C}}. \tag{6.1.2}$$

We define the topological periodic homology of the algebra using the topology given by the family of semi-norms corresponding to the control of the supremum of the function and of its derivatives of any order.

Definition 6.2. Let A be an algebra over a field k being the real or complex numbers, endowed with a family of semi-norms which makes it a Fréchet space (e.g. $\mathcal{C}^\infty(X_{sm},\mathbb{C})$ for X_{sm} a smooth variety). We define the *projective tensor product* of A with itself as

$$A \otimes_\pi A = A \otimes_k A \tag{6.2.1}$$

as a $\mathbb{C}$-algebra, but endowed with the strongest locally convex topology making it a topological vector space and making the natural application

$$A \times A \to A \otimes A \tag{6.2.2}$$

continuous. This space is often not complete, and we will denote its completion by

$$A \hat{\otimes}_\pi A = \overline{A \otimes_\pi A}, \tag{6.2.3}$$

More generally we will denote

$$A^{\hat{\otimes} n} = A \hat{\otimes}_\pi \dots \hat{\otimes}_\pi A, \tag{6.2.4}$$

for the projective tensor product of n copies of the algebra A.

Definition 6.3. The topological periodic complex associated to the algebra A is then defined as

$$\begin{tikzcd}
& {} \arrow[d, "b"] & {} \arrow[d, "b"] & {} \arrow[d, "b"] & {} \arrow[d, "b"] & {} \arrow[d, "b"] \\
{} & A^{\hat{\otimes}5} \arrow[l, "B"'] \arrow[d, "b"] & A^{\hat{\otimes}4} \arrow[l, "B"'] \arrow[d, "b"] & A^{\hat{\otimes}3} \arrow[l, "B"'] \arrow[d, "b"] & A^{\hat{\otimes}2} \arrow[l, "B"'] \arrow[d, "b"] & A \arrow[l, "B"'] \\
{} & A^{\hat{\otimes}4} \arrow[l, "B"'] \arrow[d, "b"] & A^{\hat{\otimes}3} \arrow[l, "B"'] \arrow[d, "b"] & A^{\hat{\otimes}2} \arrow[l, "B"'] \arrow[d, "b"] & A \arrow[l, "B"'] & \\
{} & A^{\hat{\otimes}3} \arrow[l, "B"'] \arrow[d, "b"] & A^{\hat{\otimes}2} \arrow[l, "B"'] \arrow[d, "b"] & A \arrow[l, "B"'] & & \\
{} & A^{\hat{\otimes}2} \arrow[l, "B"'] \arrow[d, "b"] & {\color{red}A} \arrow[l, "B"'] & & & \\
{} & A \arrow[l, "B"'] & & & &
\end{tikzcd} \tag{6.3.1}$$

,

the red A still being in position $(0,0)$. The topological periodic homology of A, denoted $HP_{top,n}(A)$ is then defined as the cohomology of the total complex associated to this periodic bicomplex. The λ-operations are still defined and give a splitting of the topological periodic homology. We define the topological cyclic homology $HC_{top,n}(A)$ the same way as before from the topological bicomplex.

This morphism π_X is in fact really useful the following proposition.

Proposition 6.4. *Let $X_{\mathbb{C}}$ be a smooth projective complex variety.*

(1) *The map π_X^* induced by π_X on periodic cyclic homology is the composition of two isomorphisms as follows, for any integers j and n*

$$\begin{tikzcd} HP_n^{(j)}(X_{\mathbb{C}}) \arrow[rr, "\pi_X^*"] \arrow[dr, "\zeta", "\simeq"'] & & HP_{top,n}^{(j)}(\mathcal{C}^\infty(X_{sm},\mathbb{C})) \\ & H_B^{2j-n}(X(\mathbb{C}),\mathbb{C}) \arrow[ur, "\simeq"] & \end{tikzcd} \tag{6.4.1}$$

and therefore the map π_X^* is itself an isomorphism.

(2) *If the pair of integers (n,j) is such that $n \geq 2\dim(X_{\mathbb{C}})$, and $n \leq 2j \leq 2n$, then the first statement is true at the level of cyclic homology, that is*

$$HC_n^{(j)}(X_{\mathbb{C}}) \simeq H_B^{2j-n}(X(\mathbb{C}),\mathbb{C}) \simeq HC_{top,n}^{(j)}(\mathcal{C}^\infty(X_{sm},\mathbb{C})). \tag{6.4.2}$$

Remark 6.5. The isomorphism

$$H_B^{2j-n}(X(\mathbb{C}),\mathbb{C}) \simeq HP_{top,n}^{(j)}(\mathcal{C}^\infty(X_{sm},\mathbb{C})) \tag{6.5.1}$$

is in fact a refinement of Connes' theorem, stated below, adding precision about the λ-decomposition.

Theorem 6.6. (Connes) *Let X_{sm} be a smooth manifold, then there is an isomorphism*

$$HP_{top,n}(\mathcal{C}^\infty(X_{sm},\mathbb{C})) \simeq \bigoplus_{i\in\mathbb{Z}} H_{dR}^{2i-n}(X_{sm},\mathbb{C}). \tag{6.6.1}$$

Let us prove proposition 6.4.

Proof. To prove these assertions, we will use Čech cohomology to compute the groups. We begin by considering an affine covering of $X_{\mathbb{C}}$ by Zariski open sets, $\mathcal{U} = (U_i)_{i\in I}$. The morphism π_X is a morphism from an affine scheme to a projective scheme, and it is therefore affine, that is to say the open sets $\mathcal{V} = (V_i)_{i\in I} = (\pi_X^{-1}(U_i))_{i\in I}$ form an affine covering of the scheme $\mathrm{Spec}(\mathcal{C}^\infty(X_{sm},\mathbb{C}))$. Using the Hochschild-Kostant-Rosenberg theorem 3.11 as we did in section 3, and section III of [Ha], we can replace the injective resolution of the algebraic de Rham complex by the Čech bicomplex, that is

$$\check{C} = \left(\check{C}^{p,q} = C^q(\mathcal{U},\Omega^p_{X_{\mathbb{C}}}),\partial,\delta\right), \tag{6.6.2}$$

where ∂ is the usual de Rham differential on $\Omega^p_{X_{\mathbb{C}}}$, extended to the Čech complex, and δ is the Čech coboundary. With this bicomplex, the periodic cyclic homology is the cohomology of the total cochain complex as follows

$$HP_n^{(j)}(X_{\mathbb{C}}) = H^{-n}\left(\bigoplus_{p+q=*} \check{C}^{p+2j,q}, \partial+\delta\right). \tag{6.6.3}$$

Then a class ω in $HP_n^{(j)}(X_{\mathbb{C}})$ can be represented by a cocycle

$$\omega = \sum_{p+q=-n} \omega_{p,q}, \text{ with } \omega_{p,q} \in C^q(\mathcal{U},\Omega^{p+2j}_{X_{\mathbb{C}}}). \tag{6.6.4}$$

The image of this class under π_X^* is therefore represented by the corresponding cocyle for the affine covering $\mathcal{V}$. On an affine open set of the form $U = U_{i_0} \cap \cdots \cap U_{i_k}$, the map π_X^* is the same as the inclusion

$$\Gamma(U, \Omega^p_{X_{\mathbb{C}}}) \xrightarrow{i} \mathcal{C}^\infty(U, \Lambda^{p,0} T^*_{\mathbb{C}}) \tag{6.6.5}$$

of the algebraic sections of the sheaf $\Omega^p_{X_{\mathbb{C}}}$ into the space of smooth sections of the vector bundle $\Lambda^{p,0} T^*_{\mathbb{C}}$ of complex differential forms of type $(p, 0)$ on U (the bundle $T^*_{\mathbb{C}}$ is the usual cotangent bundle on U). This inclusion gives in particular a morphism between two resolutions of the constant sheaf $\underline{\mathbb{C}}$, for the usual topology. The first resolution being given by the holomorphic differential forms on $X(\mathbb{C})$

$$0 \to \underline{\mathbb{C}} \to \Omega^0_{X(\mathbb{C})} \xrightarrow{\partial} \Omega^1_{X(\mathbb{C})} \xrightarrow{\partial} \dots, \tag{6.6.6}$$

and the second is given by the de Rham complex of sheaves of smooth differential forms, that is to say

$$0 \to \underline{\mathbb{C}} \to \mathcal{C}^\infty(\bullet, \Lambda^0 T^*_{\mathbb{C}}) \xrightarrow{d} \mathcal{C}^\infty(\bullet, \Lambda^1 T^*_{\mathbb{C}}) \xrightarrow{d} \dots, \tag{6.6.7}$$

where $d = \partial + \bar{\partial}$ This shows that the class of ω in the Betti (de Rham) cohomology is exactly the same as the class of $\pi_X^*(\omega)$ which is represented by the Čech cocycle $i(\omega)$ in the Čech bicomplex associated to $\mathcal{V}$, that is

$$\check{C}^{p,q} = C^q(\mathcal{V}, \mathcal{C}^\infty(\bullet, \Lambda^p T^*_{\mathbb{C}})). \tag{6.6.8}$$

As the covering $\mathcal{V}$ is made of affine open sets, this bicomplex has exact columns. From this we deduce that the cocycle $i(\omega)$ is cohomologous to a global section, that is to say it can be represented by a closed differential form of degree $2j - n$. This shows that we get the factorization of π_X^* as wanted, in two isomorphims. For the second assertion, we have for $n \geq 2 \dim X$ the following decomposition

$$HC_{top,n}(\mathcal{C}^\infty(X(\mathbb{C}), \mathbb{C})) = \bigoplus_{j \geq n/2} H_B^{2j-n}(X(\mathbb{C}), \mathbb{C}), \tag{6.6.9}$$

this is a classic variation of Connes' theorem about periodic homology, see [Lod]. The assertion follows from the splitting of the cyclic homology thanks to the λ-decomposition, using theorem 5.6. ☐

This proposition is important to understand the behaviour of the real periodic homology.

6.7. **Real periodic homology.** To define the real periodic homology, we need the notion of *derived pullback* of two complexes.

Definition 6.8. Let A, B, and C three cochain complexes with two morphisms $f : A \to C$ and $g : B \to C$. The derived pullback of the diagram is defined by:

$$A \times_C B = \mathrm{Cone}\left(A \oplus B \xrightarrow{f-g} C\right)[1] \tag{6.8.1}$$

The two natural projections give two maps from $A \times_C B$ to A and B such that the diagram

$$\begin{array}{ccc} A \times_C B & \xrightarrow{f'} & B \\ \Big\downarrow{\scriptstyle g'} & & \Big\downarrow{\scriptstyle g} \\ A & \xrightarrow{f} & C \end{array} \tag{6.8.2}$$

commutes up to canonical homotopy. The definition gives also a short exact sequence

$$0 \to \mathrm{Cone}\left(A \xrightarrow{f} C\right)[1] \to A \times_C B \xrightarrow{f'} B \to 0, \tag{6.8.3}$$

and another, switching the roles of A and B. With this sequence, we see that f' is a quasi-isomorphism if and only if f is one.

The real periodic homology is then defined as follows.

Definition 6.9. The chain complex $PC^*_{real}(X_{\mathbb{C}})$ is defined as the derived pullback of the diagram

$$\begin{tikzcd} & PC^*(X_{\mathbb{C}}) \arrow[d, "\pi_X^*"] \\ PC^*_{top}(\mathcal{C}^\infty(X_{sm},\mathbb{R})) \arrow[r, "(2i\pi)^{\Theta_0}"] & PC^*_{top}(\mathcal{C}^\infty(X_{sm},\mathbb{C})), \end{tikzcd} \tag{6.9.1}$$

where the morphism Θ_0 is defined as the "generator of the λ-operations", that is Θ_0 is equal to j on the j-th component of PC^*, with the λ-decomposition. The completed diagram is the following:

$$\begin{tikzcd} PC^*_{real}(X_{\mathbb{C}}) \arrow[r, "\tau"] \arrow[d] & PC^*(X_{\mathbb{C}}) \arrow[d, "\pi_X^*"] \\ PC^*_{top}(\mathcal{C}^\infty(X_{sm},\mathbb{R})) \arrow[r, "(2i\pi)^{\Theta_0}"] & PC^*_{top}(\mathcal{C}^\infty(X_{sm},\mathbb{C})). \end{tikzcd} \tag{6.9.2}$$

Moreover, the morphisms are compatible with the λ-decomposition and we can therefore obtain such a decomposition for HP^{real}_n.

Proposition 6.10. *As π_X^* is an isomorphism at the level of cohomology groups, the maps τ and $(2i\pi)^{\Theta_0}$ have the same effect of the cohomology. Therefore, we have a commutative diagram*

$$\begin{tikzcd} HP^{real,(j+1)}_{n+2}(X_{\mathbb{C}}) \arrow[r, "S"] \arrow[d, "\tau"] & HP^{real,(j)}_n(X_{\mathbb{C}}) \arrow[d, "\tau"] \\ HP^{(j+1)}_{n+2}(X_{\mathbb{C}}) \arrow[r, "(2i\pi)^{-1}S"] & HP^{(j)}_n(X_{\mathbb{C}}) \end{tikzcd} \tag{6.10.1}$$

Example 6.11. Suppose $X_{\mathbb{C}}$ is the scheme $\mathrm{Spec}(\mathbb{C})$, which has a single point. In that case, we have $HP^j_n = \mathbb{C}$ if $n = 2j$ and 0 else. Therefore the map $\tau : HP^{real,(j)}_n(X_{\mathbb{C}}) \to HP^{(j)}_n(X_{\mathbb{C}})$ is the multiplication by $(2i\pi)^j$ from $\mathbb{R}$ to $\mathbb{C}$.

Once again, the λ-decomposition of the real periodic homology can be computed in terms of Betti cohomology, using the morphism ζ from proposition 6.4.

Proposition 6.12. *The isomorphism ζ gives an isomorphism*

$$\zeta : HP^{real,(j)}_n(X_{\mathbb{C}}) \xrightarrow{\simeq} H^{2j-n}_B\left(X(\mathbb{C}), \mathbb{R}(j)\right),$$

such that the following diagram commutes

$$\begin{tikzcd} HP^{real,(j)}_n(X_{\mathbb{C}}) \arrow[r, "\zeta"] \arrow[d, "\tau"] & H^{2j-n}_B\left(X(\mathbb{C}), \mathbb{R}(j)\right) \arrow[d, "\iota"] \\ HP^{(j)}_n(X_{\mathbb{C}}) \arrow[r, "\zeta"] & H^{2j-n}_B\left(X(\mathbb{C}), \mathbb{C}\right). \end{tikzcd} \tag{6.12.1}$$

7. The archimedean cyclic homology

The homology theory that really interests us is the archimedean cyclic homology of schemes, which is a new definition generalizing the results of [Den].

7.1. **Definition.** Let $X_{\mathbb{C}}$ be a smooth projective complex variety, of dimension d.

Definition 7.2. The archimedean cyclic homology of the variety X is defined as

$$HC_n^{ar}(X_{\mathbb{C}}) = \mathbb{H}^{-n}\left(X_{\mathbb{C}}, \mathrm{Cone}\left(NC^*(X_{\mathbb{C}}) \oplus PC^*_{real}(X_{\mathbb{C}}) \xrightarrow{\beta} PC^*(X_{\mathbb{C}})\right)[2]\right) \tag{7.2.1}$$

the map β being defined by

$$\beta(\omega, \alpha) = I(\omega) - \tau(\alpha). \tag{7.2.2}$$

The complex of which we take the hypercohomology is in fact the derived pullback of the following diagram

$$\begin{tikzcd} & PC^*_{real}(X_{\mathbb{C}}) \arrow[d, "\tau"] \\ NC^*(X_{\mathbb{C}}) \arrow[r, "I"] & PC^*(X_{\mathbb{C}}). \end{tikzcd} \tag{7.2.3}$$

Proposition 7.3. *There is a long exact sequence*
(7.3.1)

$$\cdots \to HP^{real}_{n+2}(X_{\mathbb{C}}) \xrightarrow{S\circ\tau} HC_n(X_{\mathbb{C}}) \to HC_n^{ar}(X_{\mathbb{C}}) \to HP^{real}_{n+1}(X_{\mathbb{C}}) \xrightarrow{S\circ\tau} HC_{n-1}(X_{\mathbb{C}}) \to \dots.$$

Proof. The short exact sequence 6.8.3 is here of the form

$$0 \to \mathrm{Cone}\,(I) \to \mathrm{Cone}\,(\beta) \to PC^*_{real}(X_{\mathbb{C}})[-1] \to 0. \tag{7.3.2}$$

But as there is a short exact sequence of total complexes

$$0 \to NC^*(X_{\mathbb{C}}) \xrightarrow{I} PC^*(X_{\mathbb{C}}) \to CC^*(X_{\mathbb{C}})[-2] \to 0, \tag{7.3.3}$$

the hypercohomology of the first term of 7.3.2 is the cyclic homology (up to a shift), namely

$$\mathbb{H}^{-n}\left(X_{\mathbb{C}}, \mathrm{Cone}\,(I)\right) = HC_{n-2}(X_{\mathbb{C}}). \tag{7.3.4}$$

Taking the hypercohomology of 7.3.2, we get the sequence we were looking for. □

7.4. **Agreement with Deligne cohomology.** Just as we did above in the case when ν was a complex place of the field k, our goal is now to compare the archimedean homology with Deligne cohomology in order to obtain a suitable space on which it will be convenient to express the L-functions as a regularized determinant.

Proposition 7.5. *The sequence of proposition 7.3 splits into sequences of the form*

$$\cdots \to HP^{real,(j+1)}_{n+2}(X_{\mathbb{C}}) \xrightarrow{S\circ\tau} HC_n^{(j)}(X_{\mathbb{C}}) \to HC_n^{ar,(j)}(X_{\mathbb{C}}) \to HP^{real,(j+1)}_{n+1}(X_{\mathbb{C}}) \xrightarrow{S\circ\tau} \dots. \tag{7.5.1}$$

Note that in this sequence, the value of j is kept fixed, whereas the values of n decrease.

As in section 5, we can split these long exact sequences into short ones when the pair (n, j) is in the set E_d (see 5.8).

Proposition 7.6. *For a pair of integers $(n, j) \in E_d$, there is a short exact sequence*

$$0 \to HP^{real,(j+1)}_{n+2}(X_{\mathbb{C}}) \xrightarrow{S\circ\tau} HC_n^{(j)}(X_{\mathbb{C}}) \to HC_n^{ar,(j)}(X_{\mathbb{C}}) \to 0. \tag{7.6.1}$$

Proof. For a such pair (n,j), recall that we have $(2j-n, j-n) \in A_d$, and thus we can use the short sequence 5.13

$$0 \to H_B^{2j-n}(X(\mathbb{C}), \mathbb{R}(j+1)) \to \tilde{H}_{\mathcal{D}}^{2j-n}(X_{\mathbb{C}}, \mathbb{R}(j+1)) \to H_{\mathcal{D}}^{2j-n+1}(X_{\mathbb{C}}, \mathbb{R}(j+1)) \to 0. \tag{7.6.2}$$

Recall also that we have isomorphisms 5.6

$$HC_n^{(j)}(X_{\mathbb{C}}) \simeq \tilde{H}_{\mathcal{D}}^{2j-n+1}(X_{\mathbb{C}}, \mathbb{R}(j+1)). \tag{7.6.3}$$

Now, combining this and propositions 6.12 and 6.10, we can identify the maps

$$HP_{n+2}^{real,(j+1)}(X_{\mathbb{C}}) \xrightarrow{S\circ\tau} HC_n^{(j)}(X_{\mathbb{C}}) \tag{7.6.4}$$

and

$$H_B^{2j-n}(X(\mathbb{C}), \mathbb{R}(j+1)) \xrightarrow{\subset} \tilde{H}_{\mathcal{D}}^{2j-n+1}(X_{\mathbb{C}}, \mathbb{R}(j+1)), \tag{7.6.5}$$

which is injective and gives the splitting we were looking for. □

From this we get an isomorphism for the archimedean homology and the fact several groups must be trivial.

Corollary 7.7. *For $(n,j) \in E_d$, we have an isomorphism*

$$H_{\mathcal{D}}^{2j-n+1}(X_{\mathbb{C}}, \mathbb{R}(j+1)) \simeq HC_n^{ar,(j)}(X_{\mathbb{C}}). \tag{7.7.1}$$

Moreover, if the pair (n,j) is not in E_d with $n \geq 0$, then $HC_n^{ar,(j)}(X_{\mathbb{C}}) = 0$.

Proof. The isomorphism in the case $(n,j) \in E_d$ follows from the comparison of the two sequences used in the previous proposition, 5.13 and 7.6.1. To prove the groups are zero when (n,j) is not in E_d, we will use a result from [We2] (proposition 3.1), which states that $HC_n^{(j)}(X_{\mathbb{C}})$ is zero whenever j is strictly lower than $n/2$, and the same is also true for $2j - n > 2d$. Then, proposition 6.12 shows that we have

$$HP_{n+2}^{real,(j+1)}(X_{\mathbb{C}}) = 0, \text{ and } HP_{n+1}^{real,(j+1)}(X_{\mathbb{C}}) = 0, \tag{7.7.2}$$

whenever $2j - n > 2d$, or $2j < n-1$. With these two facts, we can apply the sequence 7.5 to get that the groups $HC_n^{ar,(j)}(X_{\mathbb{C}})$ are 0 if $2j - n > 2d$ or $2j < n-1$. Only one case remains, the limit case $n = 2j+1$. In this case, $2(j+1) - (n+1) = 0$ and therefore

$$HP_{n+1}^{real,(j+1)}(X_{\mathbb{C}}) \simeq H_B^0(X(\mathbb{C}), \mathbb{R}(j+1)) \neq 0. \tag{7.7.3}$$

but the map of the long exact sequence 7.5 is then the same as

$$H_B^0(X(\mathbb{C}), \mathbb{R}(j+1)) \to \tilde{H}_{\mathcal{D}}^{2j-n}(X_{\mathbb{C}}, \mathbb{R}(j+1)) \simeq HC_{n-1}^{(j)}(X_{\mathbb{C}}), \tag{7.7.4}$$

which is injective, because its kernel lies in $H_B^0(X(\mathbb{C}), \mathbb{R}(j+1)) \cap F^{j+1}$ which is 0. Therefore we can still use the long exact sequence to get the result. □

7.8. **The real archimedean homology.** In this section, we will extend this definition to the case of a real algebraic variety. In what follows, let $X_{\mathbb{R}}$ be a smooth projective variety over $\mathbb{R}$, of dimension d. We will denote $X_{\mathbb{C}} = X_{\mathbb{R}} \times_{\operatorname{Spec}\mathbb{R}} \operatorname{Spec}\mathbb{C}$ the complex variety obtained by extension of scalars from $X_{\mathbb{R}}$. The variety $X_{\mathbb{R}}$ has a cyclic homology theory as developed over any field.

Lemma 7.9. *For every n, we have*

$$HC_n(X_{\mathbb{C}}) \simeq HC_n(X_{\mathbb{R}}) \otimes_{\mathbb{R}} \mathbb{C}. \tag{7.9.1}$$

Proof. The result follows from the expression of the cyclic homology in terms of product of cohomology of the sheaves of Kähler differentials on $X_{\mathbb{R}}$ (which can be found in [We2], proposition 4.1), as done in section 4.32, and from the fact that these cohomology groups behave well with the tensor products in the following way

$$H^q(X_{\mathbb{C}}, \Omega^p_{X_{\mathbb{C}}}) \simeq H^q(X_{\mathbb{R}}, \Omega^p_{X_{\mathbb{R}}}) \otimes_{\mathbb{R}} \mathbb{C}. \tag{7.9.2}$$

□

Definition 7.10. From this lemma, we can see that $HC_n(X_{\mathbb{C}})$ comes with a natural anti-linear involution which we will denote F:

$$F = id \otimes_{\mathbb{R}} {}^{-}, \tag{7.10.1}$$

where $^{-}$ is the complex conjugation on $\mathbb{C}$.

Remark 7.11. It is important to note that this involution preserves the Hodge spaces

$$H^{p,q} = H^q(X_{\mathbb{R}}, \Omega^p_{X_{\mathbb{R}}}). \tag{7.11.1}$$

Moreover the image of the map

$$HP^{real,(j+1)}_{n+2}(X_{\mathbb{C}}) \xrightarrow{S\circ\tau} HC^{(j)}_n(X_{\mathbb{C}}) \tag{7.11.2}$$

is invariant under F, as we can see thanks to the identification of this map with the following

$$H^{2j-n}_B(X(\mathbb{C}), \mathbb{R}(j+1)) \xrightarrow{\subset} \tilde{H}^{2j-n+1}_{\mathcal{D}}(X_{\mathbb{C}}, \mathbb{R}(j+1)). \tag{7.11.3}$$

Therefore, the map F descends to the quotient $HC^{ar,(j)}_n(X_{\mathbb{C}})$.

Definition 7.12. The archimedean cyclic homology of the variety $X_{\mathbb{R}}$ is defined as the group of elements of the archimedean cyclic homology of $X_{\mathbb{C}}$ fixed under the action of the involution F, that is to say

$$HC^{ar}_n(X_{\mathbb{R}}) = HC^{ar}_n(X_{\mathbb{C}})^{F=id}, \tag{7.12.1}$$

this can be written in the equivalent form

$$HC^{ar}_n(X_{\mathbb{R}}) = HC_n(X_{\mathbb{R}})/(HC_n(X_{\mathbb{R}}) \cap \mathrm{Im}(S\circ\tau)). \tag{7.12.2}$$

The most important thing to note is the parallel with the definition of the Deligne cohomology in the real case, taking the fixed points under the action of the de Rham conjugation, as in 5.4. Note that the λ-decomposition still exists for these groups, and will be denoted as before.

Exactly like we did in the complex case, we get the following correspondence with Deligne cohomology.

Proposition 7.13. *Recall that the dimension of $X_{\mathbb{R}}$ is denoted d. For any pair of integers $(n, j) \in E_d$, we have an isomorphism*

$$HC^{ar,(j)}_n(X_{\mathbb{R}}) \simeq H^{2j-n+1}_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(j+1)), \tag{7.13.1}$$

and $HC^{ar,(j)}_n(X_{\mathbb{R}}) = 0$ if the pair (n, j) is not in E_d, and $n \geq 0$.

Proof. The same proof as in the complex case 7.7 still works here. □

8. The main theorem

We now have everything we need to state and prove the main theorem of this mémoire.

8.1. **Preliminaries: generalized products.** To see what the true utility of the archimedean homology developed through this mémoire, we need to define the L-function associated to a Hodge structure over one of the fields $\mathbb{R}$ or $\mathbb{C}$. The L-functions are defined as a product of shifted Γ functions. We will try to express them as determinant over certain infinite dimensional spaces. Thus, we need a method to multiply infinite families of complex numbers.

Definition 8.2. Given (λ_i, α_i) a sequence of complex numbers λ_i, with a choice of arguments α_i such that only finitely many λ_i are zero, we say that the product

$$\mathop{\prod\!\!\!\!\!\!/}_{i} \lambda_i = \mathop{\prod\!\!\!\!\!\!/}_{i} (\lambda_i, \alpha_i) \tag{8.2.1}$$

exists if the following holds: Let $N \gg 0$ be such that $\lambda_i \neq 0$ for all $i \geq N$. Then the Dirichlet series

$$\sum_{i>N} \lambda_i^{-s} = \sum_{i>N} |\lambda_i|^{-s} e^{-is\alpha_i} \tag{8.2.2}$$

converges absolutely for $\mathrm{Re}(s) \gg 0$, and has an analytic continuation to a holomorphic function $\zeta_N(s)$ for $\mathrm{Re}(s) > -\varepsilon$, for some $\varepsilon > 0$. When this is true (independently of N), we define

$$\mathop{\prod\!\!\!\!\!\!/}_{i} (\lambda_i, \alpha_i) = \left(\prod_{i \leq N} \lambda_i \right) \exp\left(-\zeta_N'(0)\right) \tag{8.2.3}$$

Remark 8.3. The following properties insure the definition is suitable in order to define determinants of linear maps on infinite dimensional spaces.

(1) For finite sequence (or sequence with finitely many complex numbers not equal to 1), then the product has the same value as the usual product.
(2) Changing a finite number of arguments α_i does not change the result, and it is also independent of the order of multiplication (that is to say invariant under the action of a bijection of the integers).
(3) Given a partition of the integers $\mathbb{N} = M \amalg M'$ such that
$$\mathop{\prod\!\!\!\!\!\!/}_{i \in M} \lambda_i \text{ and } \mathop{\prod\!\!\!\!\!\!/}_{i \in M'} \lambda_i$$
exist, then the equality
$$\mathop{\prod\!\!\!\!\!\!/}_{i \in \mathbb{N}} \lambda_i = \left(\mathop{\prod\!\!\!\!\!\!/}_{i \in M} \lambda_i \right) \cdot \left(\mathop{\prod\!\!\!\!\!\!/}_{i \in M'} \lambda_i \right)$$
holds.
(4) If λ is a complex number with an argument α, then for N sufficiently large, we have
$$\mathop{\prod\!\!\!\!\!\!/}_{i} (\lambda\lambda_i, \alpha + \alpha_i) = \lambda^{N + \zeta_N(0)} \mathop{\prod\!\!\!\!\!\!/}_{i} (\lambda_i, \alpha_i).$$

From this we can define the regularized determinant of a linear map.

Definition 8.4. Let Θ be an endomorphism of a complex vector space of countable dimension V, such that

(1) V is a direct sum of spaces stable under the action of Θ.
(2) the eigenvalues λ of Θ occur with finite multiplicity $m(\lambda)$.

(3) the eigenvalues λ_i are given a choice of argument α_i such that their regularized product exists.

In that case, the determinant of the operator Θ is defined as

$$\det\nolimits_\infty \Theta = \prod_i\!\!\!\!\!\!/\;\;(\lambda_i, \alpha_i). \tag{8.4.1}$$

The previous remark gives the following.

Remark 8.5. (1) The value of the determinant is independent from the ordering of the eigenvalues.

(2) For a finite dimensional space V, the regularized determinant agrees with the usual determinant.

(3) Given a commutative diagram with exact lines

$$\begin{tikzcd} 0 \arrow[r] & V' \arrow[r] \arrow[d, "\Theta'"] & V \arrow[r] \arrow[d, "\Theta"] & V'' \arrow[r] \arrow[d, "\Theta''"] & 0 \\ 0 \arrow[r] & V' \arrow[r] & V \arrow[r] & V'' \arrow[r] & 0, \end{tikzcd} \tag{8.5.1}$$

assuming $\det_\infty \Theta'$ and $\det_\infty \Theta''$ are defined, then $\det_\infty \Theta$ is too, and satisfies, for the obvious choices of arguments,

$$\det\nolimits_\infty \Theta = \det\nolimits_\infty \Theta' \cdot \det\nolimits_\infty \Theta''. \tag{8.5.2}$$

(4) For a complex number λ with an argument α, and N sufficiently large, we have

$$\det\nolimits_\infty \lambda\Theta = \lambda^{N+\zeta_N(0)} \det\nolimits_\infty \Theta.$$

The first application is a rewriting of the Γ function as a regularized product.

Definition 8.6. Let Γ be the usual function defined as

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \tag{8.6.1}$$

at first for $\mathrm{Re}(z) > 0$. This function is holomorphic on the right half of the complex plane, and can be extended to a meromorphic function defined on the whole complex plane, with simple poles at points $z \in \{0, -1, -2, \dots\}$ and such that

$$\mathrm{Res}_{z=-n} \Gamma(z) = \frac{(-1)^n}{n!}, \tag{8.6.2}$$

for all integers $n \geq 0$. Using this function we define the complex and real Γ-functions, as in [Ser]

$$\Gamma_{\mathbb{C}}(z) = (2\pi)^{-z}\, \Gamma(z), \tag{8.6.3}$$

and

$$\Gamma_{\mathbb{R}}(z) = 2^{-\frac{1}{2}} \pi^{-\frac{z}{2}} \Gamma\left(\frac{z}{2}\right). \tag{8.6.4}$$

Proposition 8.7. *The inverse of these functions, which are everywhere holomorphic, can be written as regularized products as follows, for any complex number z,*

$$\left(\frac{1}{\sqrt{2\pi}} \Gamma(z)\right)^{-1} = \prod_{m=0}^{\infty}\!\!\!\!\!\!/\;\;(m+z), \tag{8.7.1}$$

$$\Gamma_{\mathbb{C}}(z)^{-1} = \prod_{m=0}^{\infty}{}' \frac{m+z}{2\pi}, \tag{8.7.2}$$

$$\Gamma_{\mathbb{R}}(z)^{-1} = \prod_{m=0}^{\infty}{}' \frac{2m+z}{2\pi}, \tag{8.7.3}$$

taking the arguments of the factors to be in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$*, for* m *sufficiently large.*

Proof. Since

$$\Gamma_{\mathbb{C}}(z+1) = \frac{z}{2\pi}\Gamma_{\mathbb{C}}(z), \text{ and } \Gamma_{\mathbb{R}}(z+2) = \frac{z}{2\pi}\Gamma_{\mathbb{R}}(z),$$

we can assume that $z \geq 0$ and thus we can take the argument of every factor in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. We define the Hurwitz zeta function

$$\zeta(s,z) = \sum_{m=0}^{\infty} \frac{1}{(m+z)^s}, \text{ for } \mathrm{Re}(z) > 0, \tag{8.7.4}$$

is defined for $\mathrm{Re}(s) > 1$, and has an analytic continuation to $\mathbb{C} \setminus \{1\}$. For $s = 0$, we have

$$\zeta(0,z) = \frac{1}{2} - z, \text{ and } (\partial_s\zeta)(0,z) = \log\Gamma(z) - \frac{1}{2}\log(2\pi), \tag{8.7.5}$$

with a suitable branch of $\log\Gamma$. From this we go back to the definition of the right-hand product.

$$\prod_{m=0}^{\infty}{}' \frac{1}{m+z} = \exp(-\partial_z\zeta(0,z)), \tag{8.7.6}$$

because the Hurwitz zeta function agrees with the corresponding Dirichlet series for $s = 0$. Thus, as we know the value of $-\partial_z\zeta(0,z)$, taking the exponential gives the result. □

8.8. **Definitions and statement.** The general framework to define the L-function is the abelian category of Hodge structures over $\mathbb{R}$ or $\mathbb{C}$. The cohomology rings of a variety are a special case of Hodge structure thanks to Hodge theory as we will see later.

Definition 8.9. A *Hodge structure* over $\mathbb{C}$ is a finite dimensional bigraded $\mathbb{C}$-vector space $H = \bigoplus_{p,q} H^{p,q}$, with a $\mathbb{C}$-antilinear involution c such that $c(H^{p,q}) = H^{q,p}$, and the inclusion of $H_{\mathbb{R}} = H^{c=id}$ into H induces an isomorphism $H_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \simeq H$. A *Hodge structure* over $\mathbb{R}$ is a Hodge structure over $\mathbb{C}$ together with a $\mathbb{C}$-linear involution F_∞ which commutes with c and is such that $F_\infty(H^{p,q}) = H^{q,p}$.

A good way to get such structures is to endow the cohomology of complex manifold with coefficient in $\mathbb{C}$ with the involutions we seek. This is possible when the considered manifold is a *Kähler manifold*.

Definition 8.10. Let Y be a complex manifold, endowed with a hermitian metric h. Then the *associated* 2*-form* of the metric h is the 2-form ω such that

$$\omega(u,v) = \mathrm{Re}(h(iu,v)) = \mathrm{Im}(h(u,v)), \tag{8.10.1}$$

for u and v two tangent vectors. We say that the manifold Y is a *Kähler manifold* if the $(1,1)$-form ω is closed.

Example 8.11. The manifold $\mathbb{C}$ with its usual hermitian metric is a Kähler manifold, and more generally, submanifold of a Kähler manifold is Kähler for the induced metric, therefore, any open subset of $\mathbb{C}^n$ is a Kähler manifold.

Proposition 8.12. *The complex manifolds associated to the complex projective spaces, $\mathbb{P}^d(\mathbb{C})$, can be endowed with an hermitian metric, called the* Fubini-Study *metric.*

Proof. We define the metric on $\mathbb{P}^d(\mathbb{C})$ by specifying the hermitian matrices on every open set of the usual covering $(U_i)_{0\leq i\leq d}$ where $U_i = \{z_i \neq 0\} \subset \mathbb{P}^d(\mathbb{C})$, in homogeneous coordinates $Z = [z_0 : z_1 : \cdots : z_d]$. On the open set U_i, isomorphic to $\mathbb{C}^d$ be normalizing the i-th coordinate to 1, we define the functions

$$\begin{aligned} h_{ij} : \mathbb{C}^d &\to \mathbb{C} \\ (z_1, \dots, z_d) &\mapsto \frac{\left(1 + \sum_{i=1}^d |z_i|^2\right) \delta_{i,j} - \overline{z_i} z_j}{\left(1 + \sum_{i=1}^d |z_i|^2\right)}. \end{aligned} \tag{8.12.1}$$

Putting these functions in a matrix, we get an hermitian matrix which defines the Fubini-Study metric on the open sets of the usual covering. It is easy to check that the matrix defines a metric on the whole space $\mathbb{P}^d(\mathbb{C})$, which is invariant by the multiplication by a complex number of norm equal to 1 in every open set (the i-th coordinate being fixed at 1). □

Proposition 8.13. *Given X a smooth projective variety over the complex numbers, then the manifold built from the complex points of X, $X(\mathbb{C})$ is a Kähler manifold, for the restriction of the* Fubini-Study metric *on $X(\mathbb{C}) \subset \mathbb{P}^d(\mathbb{C})$, for a suitable integer d.*

Thanks to Hodge theory, it is possible to get a decomposition of the cohomology of Y as a direct sum of spaces defined by the degree of the forms. First, let us define the *Hodge star operator* on the cohomology of such spaces.

Definition 8.14. The Hodge star operator $\star$ is an operator which acts on the exterior algebra of any finite dimensional vector space V endowed with an inner product. In our case we will give the definition for the cotangent bundle of a manifold. Let then M be an oriented Riemannian manifold of dimension n, with volume form $d\mathrm{Vol}$, given by the Riemannian metric. The Riemannian metric gives also rise to an inner product on the cotangent spaces T_p^*M and their exterior products, at every point p of M, and therefore on the space of global differentials k-forms $\mathcal{C}^\infty(M, \Lambda^k T^*M)$. The Hodge dual of a global k-form ζ is then defined as the unique global $(n-k)$-form satisfying

$$\alpha \wedge \star\zeta = \langle \alpha, \zeta \rangle \cdot d\mathrm{Vol}, \tag{8.14.1}$$

for any k-form α on M.

Definition 8.15. On a complex manifold Y of dimension N, it is possible to define three different Laplacian operators on the spaces of smooth forms, they are

$$\Delta_d = dd^* + d^*d, \tag{8.15.1}$$

$$\Delta_\partial = \partial\partial^* + \partial^*\partial, \text{ and} \tag{8.15.2}$$

$$\Delta_{\bar\partial} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial. \tag{8.15.3}$$

The operators with a star are defined using the *Hodge star operator*, $\star$, previously defined

$$d^* = (-1)^{Nr+1} \star d\star, \tag{8.15.4}$$

on $\mathcal{C}^\infty(Y, \Lambda^{p,q}T^*Y)$, where $r = p + q$ is the degree of the form, the other two having the same definition, replacing d with the other operators.

Theorem 8.16. *If Y is a Kähler manifold, then the three Laplacians agree up to multiplication by a scalar, more precisely*

$$\Delta_d = 2\Delta_\partial = 2\Delta_{\bar\partial}. \tag{8.16.1}$$

A form ω such that $\Delta\omega$ is zero, for any of the three Laplacians, the form is said to be harmonic

This allows to split the spaces of homogeneous harmonic forms into a direct sum, indexed by the degree of the forms.

Proposition 8.17. *If Y is a Kähler manifold, then we have for every integer n*

$$\mathcal{H}^n(Y) = \bigoplus_{p+q=n} \mathcal{H}^{p,q}(Y), \tag{8.17.1}$$

where $\mathcal{H}^n(Y)$ is the space of harmonic n-forms, and $\mathcal{H}^{p,q}(Y)$ is the space of harmonic (p,q)-forms.

Moreover, these groups can be interpreted in the compact case as Dolbeault cohomology group.

Definition 8.18. Recall the definition of the Dolbeault complexes, for Y a compact Kähler complex manifold:

$$0 \to \mathcal{C}^\infty(Y, \Lambda^{p,0}T^*Y) \xrightarrow{\bar\partial} \mathcal{C}^\infty(Y, \Lambda^{p,1}T^*Y) \xrightarrow{\bar\partial} \mathcal{C}^\infty(Y, \Lambda^{p,2}T^*Y) \xrightarrow{\bar\partial} \dots, \tag{8.18.1}$$

the Dolbeault cohomology groups are defined as

$$H^{p,q}(Y) = H^q(\mathcal{C}^\infty(Y, \Lambda^{p,q}T^*Y), \bar\partial). \tag{8.18.2}$$

Theorem 8.19. (Dolbeault) *The Dolbeault cohomology of Y in degree (p,q) is naturally isomorphic to the cohomology of the sheaf Ω_Y^p of holomorphic differential $(p,0)$-forms on Y, that is to say*

$$H^{p,q}(Y) \simeq H^q(Y, \Omega_Y^p). \tag{8.19.1}$$

Theorem 8.20. *If Y is compact and Kähler, the canonical morphism*

$$\mathcal{H}^{p,q}(Y) \to H^{p,q}(Y) \tag{8.20.1}$$

is an isomorphism. Therefore, the Dolbeault cohomology groups are exactly the kernel of the Laplacian $\Delta_{\bar\partial}$.

Proposition 8.21. *The de Rham cohomology corresponds to the harmonic forms for Δ_d, that is to say $H^m_{dR}(Y)$ is equal to the kernel of the operator Δ_d on $\mathcal{C}^\infty(Y, \Lambda^m T^*Y)$.*

Proposition 8.22. *From the previous theorem, and the equality of the Laplacians from theorem 8.16 we get the Hodge decomposition of the cohomology of the manifold Y if it is compact and Kähler:*

$$H^n(Y) = \bigoplus_{p+q=n} H^{p,q}(Y). \tag{8.22.1}$$

Moreover, we get an involution on the cohomology by considering the conjugation on the forms, that is to say, the map sending a function $f(z)$ to $\overline{f(z)}$ and the forms dz_k to $d\bar z_k$ involutively.

Proposition 8.23. *This decomposition of the cohomology groups of Y come with an involution c, which correspond to the conjugation on the space of smooth forms on Y, and this involution is such that*

$$c(H^{p,q}(Y)) = H^{q,p}(Y). \tag{8.23.1}$$

Therefore, we get a complex Hodge structure for $H^*(Y)$.

In the case where the variety comes from a real projective variety by extension of scalars, it is possible to define an involution of the manifold, which will be anti-holomorphic and commute with c. The following theorem is a kind of a "real form" of Grothendieck *GAGA theorem.*

Theorem 8.24. *There is an equivalence of categories*

$$\left\{ \begin{array}{c} \textit{Smooth projective varieties} \\ \textit{over } \mathbb{R} \end{array} \right\} \xrightarrow{\simeq} \left\{ \begin{array}{c} \textit{Analytic complex manifolds endowed} \\ \textit{with an anti-holomorphic involution} \end{array} \right\} \tag{8.24.1}$$
$$X \longmapsto X(\mathbb{C}).$$

Proposition 8.25. *Given a real smooth projective variety X, it can be seen as the zero locus of a family of homogeneous polynomials with real coefficients, in a projective space $\mathbb{P}^n$, and therefore the complex points are stable under the action of the canonical anti-holomorphic involution of $\mathbb{P}^n(\mathbb{C})$, defined by*

$$\sigma\left([z_0 : \cdots : z_n]\right) = [\bar{z_0} : \cdots : \bar{z_n}], \tag{8.25.1}$$

and thus σ gives an anti-holomorphic involution on $X(\mathbb{C})$. We denote by F_∞ the action of this involution on the cohomology of $X(\mathbb{C})$, where the action is given by push-forward of the forms.

Remark 8.26. Even though it is not obvious, the two involutions c and F_∞ do not necessarily agree. For example, on the cohomology of the point $\{*\}$, seen as a complex manifold but also the complexification of the single point over $\mathbb{R}$ as a scheme, the involution c coming from the complex stucture is the usual complex conjugation on the coefficient, whereas the one coming from the real structure must be trivial, as the diffeomorphism associated on the manifold must be trivial as there is a single point. Moreover, proposition 9.7 will give a larger family of non trivial examples where this holds.

Proposition 8.27. *Let X be a real smooth projective variety, then the cohomology of $X(\mathbb{C})$, together with the involutions c and F_∞ is a real Hodge structure.*

Definition 8.28. The L-function associated to a Hodge structure H is defined as

$$L_{\mathbb{C}}\left(H, s\right) = \prod_{p,q} \Gamma_{\mathbb{C}}\left(s - \min(p,q)\right)^{h^{p,q}}, \tag{8.28.1}$$

if the structure is complex, and

$$L_{\mathbb{R}}\left(H, s\right) = \prod_{p<q} \Gamma_{\mathbb{C}}(s-p)^{h^{p,q}} \prod_{p} \Gamma_{\mathbb{R}}(s-p)^{h^{p,+}} \Gamma_{\mathbb{R}}(s-p+1)^{h^{p,-}}, \tag{8.28.2}$$

if the structure is real, where the exponents are defined as

$$h^{p,q} = \dim_{\mathbb{C}} H^{p,q}, \text{ and } h^{p,\pm} = \dim_{\mathbb{C}} H^{p,p,F_\infty = \pm(-1)^p}, \tag{8.28.3}$$

that is to say $h^{p,\pm}$ is the complex dimension of the eigenspace of $H^{p,p}$ associated to the eigenvalue $\pm(-1)^p$ of F_∞.

Example 8.29. The easiest example of a Hodge structure is the one associated to the scheme $\operatorname{Spec}\mathbb{C}$, which is a single point. The cohomology of the corresponding manifold is $H = H^{0,0} = \mathbb{C}$, with c acting as the usual conjugation. In what follows, we will write

$$L_{\mathbb{C}}\left(H(\operatorname{Spec}\mathbb{C}), s\right) = \Gamma_{\mathbb{C}}(s). \tag{8.29.1}$$

Accordingly, as $\operatorname{Spec}\mathbb{C} = \operatorname{Spec}\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C}$, then it can also be endowed with a real structure, F_∞ being the identity on the manifold and thus on the cohomology. We then get

$$L_{\mathbb{R}}\left(H(\operatorname{Spec}\mathbb{R}), s)\right) = \Gamma_{\mathbb{R}}(s). \tag{8.29.2}$$

The work done in [Den] explains how it is possible to express the L-function associated to a Hodge structure as the determinant of an operator on a newly defined *archimedean homology* (which is not the same as the one introduced in this mémoire). The goal of this mémoire is to see how the archimedean cyclic homology is a great framework in order to do the same as Deninger.

Theorem 8.30. *Let X be a smooth, projective variety of dimension d over an algebraic number field K, and let $\nu|\infty$ be an archimedean place of K ($K_\nu = \mathbb{R}$ or $\mathbb{C}$). Let $\Theta = \Theta_0 - \Gamma$ be the operator on $HC_*^{ar}(X_\nu)$, with Θ_0 the "generator" of the λ-operations on the homology (that is to say $\Theta_0 = j$ on $HC_*^{ar,(j)}(X_\nu)$), and Γ the grading operator (that is to say $\Gamma = n$ on $HC_n(X_\nu)$). Then the action of Θ satisfies the formula*

$$\prod_{0\leq m\leq 2d} L_\nu\left(H^m(X), s\right)^{(-1)^{m+1}} = \frac{\det_\infty\left(\frac{1}{2\pi}\left(s-\Theta\right)_{|HC^{ar}_{even}(X_\nu)}\right)}{\det_\infty\left(\frac{1}{2\pi}\left(s-\Theta\right)_{|HC^{ar}_{odd}(X_\nu)}\right)}, \tag{8.30.1}$$

for all complex numbers s. The spaces labeled "even" and "odd" are defined as

$$\begin{aligned} HC^{ar}_{even}(X_\nu) &= \bigoplus_{\substack{n\equiv 0[2],\\ n\geq 0}} HC^{ar}_n(X_\nu), \text{ and} \\ HC^{ar}_{odd}(X_\nu) &= \bigoplus_{\substack{n\equiv 1[2],\\ n\geq 1}} HC^{ar}_n(X_\nu). \end{aligned} \tag{8.30.2}$$

8.31. **Proof of theorem 8.30.** Let X be a variety of dimension d as above. First, recall that we have

$$H^{ar,(j)}_n(X_\nu) = 0 \tag{8.31.1}$$

whenever $n \geq 0$ and the pair (n, j) is not in E_d, thanks to propositions 7.7 and 7.13. Therefore, it is possible to write the numerator of the right-hand side of 8.30.1 as

$$\det_\infty\left(\frac{1}{2\pi}\left(s-\Theta\right)_{|HC^{ar}_{even}(X_\nu)}\right) = \det_\infty\left(\frac{1}{2\pi}\left(s-\Theta\right)_{|\mathcal{E}}\right), \tag{8.31.2}$$

where we have defined the subspace $\mathcal{E}$ of $HC^{ar}_{even}(X_\nu)$ as

$$\mathcal{E} = \bigoplus_{\substack{n\geq 0 \text{ even}\\ (n,j)\in E_d}} HC^{ar,(j)}_n(X_\nu). \tag{8.31.3}$$

This equality holds as long as the determinant in the right-hand side is defined. Moreover, the same holds with the denominator, replacing "even" by "odd" in the formulas.

Recall the identifications of the archimedean cyclic homology with Deligne cohomology, as seen again in propositions 7.7 and 7.13, for all $(n, j) \in E_d$

$$HC^{ar,(j)}_n(X_\nu) \simeq H^{2j-n+1}_{\mathcal{D}}(X_\nu, \mathbb{R}(j+1)). \tag{8.31.4}$$

Remark 8.32. It is important to keep in mind that Deligne cohomology is not the same in the real and in the complex case, the real one being defined as the fixed points of the complex one through an involution, see 5.4.

Using proposition 5.11, we can reindex the groups used in the definition of the space $\mathcal{E}$ as

$$\begin{aligned}\mathcal{E} &= \bigoplus_{\substack{n\geq 0 \text{ even}\\(n,j)\in E_d}} HC_n^{ar,(j)}(X_\nu)\\ &= \bigoplus_{\substack{n\geq 0 \text{ even}\\(n,j)\in E_d}} H_{\mathcal{D}}^{2j-n+1}(X_\nu,\mathbb{R}(j+1))\\ &= \bigoplus_{\substack{w\geq 0 \text{ even}\\(m,w)\in A_d}} H_{\mathcal{D}}^{2j-n+1}(X_\nu,\mathbb{R}(w-m+1)).\end{aligned} \tag{8.32.1}$$

Moreover, it is easy to understand how the operator Θ acts on this space, as it is just the multiplication by m on every component indexed by a pair (m,w). We denote by M the corresponding operator. From this, we have reduced the problem to showing that for any w such that $0\leq w\leq 2d$, we have

$$L_\nu\left(H^w(X),s\right)^{-1} = \det{}_\infty\left(\frac{1}{2\pi}(s-M)_{|D_w}\right), \tag{8.32.2}$$

where the space D_w is defined as

$$D_w = \bigoplus_{m\leq w/2} H_{\mathcal{D}}^{w+1}(X_\nu,\mathbb{R}(w-m+1)). \tag{8.32.3}$$

Now to prove this, we will use results by Beilinson, which are developed again in [Den]. The main point is that the poles of the L-function associated to a Hodge structure of a variety have multiplicity that can be express in terms of dimension of Deligne cohomology groups. We partly saw this fact as theorem 5.9.

Theorem 8.33. (Beilinson, [Bei]) *Let M be a motive over a number field k (or in our case, a smooth projective variety of dimension d). Let also w be a* Hodge weight *for M, and ν be an infinite (archimedean) place for k. Then for every integer m such that $m\leq w/2$, we have*

$$\mathrm{ord}_{s=m}L_\nu\left(H^w(M),s\right)^{-1} = \dim_{\mathbb{R}} H_{\mathcal{D}}^{w+1}(M_\nu,\mathbb{R}(w+1-m)). \tag{8.33.1}$$

The Deligne cohomology is different if the place ν is real or complex, as in definition 5.4. Moreover, this formula shows every possible pole for the considered L-function.

Moreover, Deninger showed in [Den] that the L-function in the left-hand side of 8.32.2 can be express as the regularized determinant of another operator on a different archimedean homology, and therefore, it is enough to check that the eigenvalues of the two operators occur in the product with the same multiplicity, but this multiplicity is exactly defined by the previous theorem. In conclusion, we get the equality 8.32.2 as expected.

The same reasoning can be done with the denominator of 8.30.1, with the odd Hodge weights. We thus get the main theorem 8.30.

9. An example: The real and complex projective spaces

In this section, we will see how the two sides of the formula 8.30.1 agree in the case where $X=\mathbb{P}_k^d$.

9.1. **The complex case.** Recall that we know the cyclic homology of the projective spaces over any field k with 4.16:

$$HC_n(\mathbb{P}^d_k) = \begin{cases} k^{d+1} \text{ for } n \geq 0,\ n \text{ even} \\ 0 \text{ otherwise.} \end{cases} \tag{9.1.1}$$

Using the short exact sequence for the computation of archimedean homology 7.6.1 (for suitable pairs of integers $(n, j) \in E_d$),

$$0 \to HP^{real,(j+1)}_{n+2}(X_{\mathbb{C}}) \to HC^{(j)}_n(X_{\mathbb{C}}) \to HC^{ar,(j)}_n(X_{\mathbb{C}}) \to 0, \tag{9.1.2}$$

it suffices to compute the real periodic cyclic homology of $\mathbb{P}^d_{\mathbb{C}}$. In order to do that we can use the isomorphism

$$HP^{real,(j)}_n(X_{\mathbb{C}}) \simeq H^{2j-n}_B(X(\mathbb{C}), \mathbb{R}(j)), \tag{9.1.3}$$

and the well-known Betti cohomology of the projective spaces:

$$H^m_B(\mathbb{P}^d(\mathbb{C}), \mathbb{R}(j)) = \mathbb{R}(j) \text{ if } 0 \leq m \leq 2d,\ m \text{ even.} \tag{9.1.4}$$

From this we get, that the real periodic homology is trivial at odd graduations, and if n is even

$$HP^{real,(j)}_n\left(\mathbb{P}^d_k\right) = \begin{cases} \mathbb{R}(j) \text{ for } 2j - n \in \{0, 2, 4, \dots, 2d-2, 2d\} \\ 0 \text{ otherwise.} \end{cases} \tag{9.1.5}$$

Knowing the dimension of the groups $HC_n(X_{\mathbb{C}})$ to be equal to 0 or $d+1$, we obtain the λ-decomposition of the cyclic homology of $X_{\mathbb{C}}$, and thus the archimedean homology,

$$HC^{ar,(j)}_n\left(\mathbb{P}^d_{\mathbb{C}}\right) = \begin{cases} \mathbb{C}/\mathbb{R}(j+1) \text{ for } 2j - n \in \{0, 2, 4, \dots, 2d-2, 2d\} \\ 0 \text{ otherwise.} \end{cases} \tag{9.1.6}$$

The eigenvalues can be represented in a table, with n on the vertical axis and $(n-2j)/2$ on the horizontal axis. The formula used being $j - n = n/2 - (n-2j)/2$.

(9.1.7)

	0	1	$\dots$	$d-1$	d
0	0	1	$\dots$	$d-1$	d
2	-1	0	$\dots$	$d-2$	$d-1$
4	-2	-1	$\dots$	$d-3$	$d-2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$2m$	$-m$	$1-m$	$\dots$	$d-1-m$	$d-m$

Proposition 9.2. *By multiplying column by column, we can write the regularized determinant as*

$$\begin{aligned} \det\nolimits_\infty \left(\frac{1}{2\pi}(s - \Theta)_{|HC^{ar}_{even}(\mathbb{P}^d_{\mathbb{C}})} \right) &= \prod_{p=0}^{d} \coprod_{m=0}^{\infty} \left(\frac{s-(p-m)}{2\pi} \right) \\ &= \prod_{p=0}^{d} \Gamma_{\mathbb{C}}(s-p). \end{aligned} \tag{9.2.1}$$

Let us now check that this agrees with the left-hand side of formula 8.30.1. The de Rham cohomology of $\mathbb{P}^d_{\mathbb{C}}$ is as we said above

$$H^m(\mathbb{P}^d(\mathbb{C}), \mathbb{C}) = \mathbb{C} \text{ if } 0 \leq m \leq 2d,\ m \text{ even,} \tag{9.2.2}$$

with for every integer p, $H^{2p} = H^{p,p}$ the involution c being everywhere the usual conjugation. The left-hand side is thus

$$\prod_{\substack{0\leq m\leq 2d,\\ m \text{ even}}} L_{\mathbb{C}}\left(H^m\left(\mathbb{P}^d\left(\mathbb{C}\right)\right), s\right)^{+1} = \prod_{p=0}^{d} \Gamma_{\mathbb{C}}(s-p)^{+1}, \tag{9.2.3}$$

which matches perfectly what we computed earlier.

9.3. **The real case.** Knowing the complex archimedean homology, it is easy to compute the real one, using directly definition 7.12. Indeed, the involution F_∞ on the complex archimedean homology of the projective spaces depends only on j, as we know each $HC_n^{ar,(j)}$ is of the form $\mathbb{C}/\mathbb{R}(j+1)$. The involution F_∞ is induced by the usual conjugation on $\mathbb{C}$. Thus, F_∞ is trivial on $\mathbb{C}/\mathbb{R}(p)$ if p is odd, but acts as $-Id$ if p is even. The fixed points are thus either the whole group, if p is odd, or 0 if p is even.

Proposition 9.4. *The real archimedean homology of $\mathbb{P}^d_{\mathbb{R}}$ is, for d even:*

$$\begin{aligned} HC_{2m}^{ar,(j)}\left(\mathbb{P}^d_{\mathbb{R}}\right) &= \begin{cases} \mathbb{C}/\mathbb{R}(j+1) \text{ for } 2j-2m \in \{0,2,4,\dots,2d-2,2d\},\ j \text{ even},\\ 0 \text{ otherwise,}\end{cases}\\ &= \begin{cases} \mathbb{C}/\mathbb{R}(j+1) \text{ for } j \in \{m, m+2, \dots, m+d\},\ \text{if } m \text{ is even,}\\ \mathbb{C}/\mathbb{R}(j+1) \text{ for } j \in \{m+1, m+3, \dots, m+d-1\},\ \text{if } m \text{ is odd,}\end{cases}\end{aligned} \tag{9.4.1}$$

and for d odd:

$$\begin{aligned} HC_{2m}^{ar,(j)}\left(\mathbb{P}^d_{\mathbb{R}}\right) &= \begin{cases} \mathbb{C}/\mathbb{R}(j+1) \text{ for } 2j-2m \in \{0,2,4,\dots,2d-2,2d\},\ j \text{ even},\\ 0 \text{ otherwise.}\end{cases}\\ &= \begin{cases} \mathbb{C}/\mathbb{R}(j+1) \text{ for } j \in \{m, m+2, \dots, m+d-1\},\ \text{if } m \text{ is even,}\\ \mathbb{C}/\mathbb{R}(j+1) \text{ for } j \in \{m+1, m+3, \dots, m+d\},\ \text{if } m \text{ is odd,}\end{cases}\end{aligned} \tag{9.4.2}$$

To express the determinant of the operator Θ on the real archimedean homology, it is again easier to summarize the values in a table. There are, as the homology groups suggest, four cases, depending on the parity of d and m.

When d is even, the table for even m is (with $j-m$ on the horizontal axis, and $m = n/2$ on the vertical one):

(9.4.3)

	0	2	$\dots$	$d-2$	d
0	0	2	$\dots$	$d-2$	d
2	-2	0	$\dots$	$d-4$	$d-2$
4	-4	-2	$\dots$	$d-6$	$d-4$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
m	$-m$	$2-m$	$\dots$	$d-2-m$	$d-m$

For odd m, it is:

(9.4.4)

	1	3	$\dots$	$d-3$	$d-1$
1	0	2	$\dots$	$d-4$	$d-2$
3	-2	0	$\dots$	$d-6$	$d-4$
5	-4	-2	$\dots$	$d-8$	$d-6$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
m	$1-m$	$3-m$	$\dots$	$d-3-m$	$d-1-m$

Therefore, we can express the determinant of Θ on the archimedean homology as a product of shifted Γ-functions by multiplying column by column.

Proposition 9.5. *When d is even, the left-hand side of formula 8.30.1 is*

(9.5.1)

$$\begin{aligned}\det_\infty\left(\frac{1}{2\pi}(s-\Theta)_{|HC^{ar}_{even}(\mathbb{P}^d_{\mathbb{R}})}\right) &= \prod_{\substack{p=0\\ p\, even}}^{d} \mathop{\not{\prod}}_{m=0}^{\infty}\left(\frac{s-(p-2m)}{2\pi}\right)\prod_{\substack{p=1\\ p\, odd}}^{d-1} \mathop{\not{\prod}}_{m=0}^{\infty}\left(\frac{s-(p-(2m+1))}{2\pi}\right)\\ &= \prod_{\substack{p=0\\ p\, even}}^{d} \mathop{\not{\prod}}_{m=0}^{\infty}\left(\frac{s-(p-2m)}{2\pi}\right)\prod_{\substack{p=0\\ p\, even}}^{d-2} \mathop{\not{\prod}}_{m=0}^{\infty}\left(\frac{s-(p-(2m))}{2\pi}\right)\\ &= \left(\prod_{\substack{p=0,\\ p\, even}}^{d-2}\Gamma_{\mathbb{R}}(s-p)^2\right)\cdot\Gamma_{\mathbb{R}}(s-d).\end{aligned}$$

Now when the dimension d is odd, we get for even m:

(9.5.2)

	0	2	$\dots$	$d-3$	$d-1$
0	0	2	$\dots$	$d-3$	$d-1$
2	-2	0	$\dots$	$d-5$	$d-3$
4	-4	-2	$\dots$	$d-7$	$d-5$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
m	$-m$	$2-m$	$\dots$	$d-3-m$	$d-1-m$

And for odd m:

(9.5.3)

	1	3	$\dots$	$d-2$	d
1	0	2	$\dots$	$d-3$	$d-1$
3	-2	0	$\dots$	$d-5$	$d-3$
5	-4	-2	$\dots$	$d-7$	$d-5$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
m	$1-m$	$3-m$	$\dots$	$d-2-m$	$d-m$

We get then

Proposition 9.6. *When d is odd, the left-hand side of formula 8.30.1 is*

(9.6.1)

$$\begin{aligned}\det_\infty\left(\frac{1}{2\pi}(s-\Theta)_{|HC^{ar}_{even}(\mathbb{P}^d_{\mathbb{R}})}\right) &= \prod_{\substack{p=0\\ p\, even}}^{d-1} \mathop{\not{\prod}}_{m=0}^{\infty}\left(\frac{s-(p-2m)}{2\pi}\right)\prod_{\substack{p=1\\ p\, odd}}^{d} \mathop{\not{\prod}}_{m=0}^{\infty}\left(\frac{s-(p-(2m+1))}{2\pi}\right)\\ &= \left(\prod_{\substack{p=0\\ p\, even}}^{d-1} \mathop{\not{\prod}}_{m=0}^{\infty}\left(\frac{s-(p-2m)}{2\pi}\right)\right)^2\\ &= \prod_{\substack{p=0,\\ p\, even}}^{d-1}\Gamma_{\mathbb{R}}(s-p)^2.\end{aligned}$$

Let us now check that these functions agree with the L-function of the varieties.

Proposition 9.7. *The involution F_∞ induced by the complex conjugation on the manifold $\mathbb{P}^d(\mathbb{R})$ acts on its cohomology as follows. On the groups $H^{2p}(\mathbb{P}^d(\mathbb{C})) = H^{p,p}(\mathbb{P}^d(\mathbb{C})) \simeq \mathbb{C}$, F_∞ acts as the identity if p is even, and as -1 if p is odd ($p \leq d$).*

We will now prove this proposition, by giving explicit bases of the de Rham cohomology groups of these spaces. We begin by describing the volume forms on spheres of odd dimension, in order to obtain the volume forms on the projective spaces from them through a quotient.

Proposition 9.8. *Let $\mathbb{S}^{2p+1}$ be the $(2p+1)$-dimensional sphere, defined equivalently (for the topological structure) as*

$$\mathbb{S}^{2p+1} = \left\{ (x_1, x_2, \dots, x_{2p+2}) \in \mathbb{R}^{2p+2}\,,\ \sum_{j=1}^{2p+2} x_i^2 = 1 \right\}, \tag{9.8.1}$$

or as

$$\mathbb{S}^{2p+1} = \left\{ (z_1, z_2, \dots, z_{p+1}) \in \mathbb{C}^{p+1}\,,\ \sum_{j=1}^{p+1} |z_i|^2 = 1 \right\}, \tag{9.8.2}$$

the change of coordinates from one description to another is given by

$$z_k = x_{2k-1} + i x_{2k}, \tag{9.8.3}$$

for all integer k. These sphere are closed real submanifolds of $\mathbb{R}^{2p+2}$ or $\mathbb{C}^{p+1}$. Consider now the $(2p+1)$-form on $\mathbb{R}^{2p+1}$

$$\omega = \sum_{k=1}^{2p+1} (-1)^k x_k dx_1 \wedge \cdots \wedge dx_{k-1} \wedge dx_{k+1} \wedge \cdots \wedge dx_{2p+2}. \tag{9.8.4}$$

Then the volume form on the sphere $\mathbb{S}^{2p+1}$ is given by the pullback of the form ω along the inclusion $\iota : \mathbb{S}^{2p+1} \hookrightarrow \mathbb{R}^{2p+2}$, that is the form $\iota^*\omega$. It is easy to see this form is invariant by the action of $O_{2p+2}(\mathbb{R})$ and is therefore nowhere zero.

Now we must use another property of the projective spaces in order to find their volume forms from the one of the spheres we just got.

Proposition 9.9. *There is a fibration of the form*

$$\begin{tikzcd} \mathbb{S}^1 \arrow[r] & \mathbb{S}^{2p+1} \arrow[d] \\ & \mathbb{P}^p(\mathbb{C}), \end{tikzcd} \tag{9.9.1}$$

which realizes the projective spaces as quotient of the $(2p+1)$-sphere by the action of the Lie group $\mathbb{S}^1$.

Proof. Seeing the $(2p+1)$-sphere embedded in $\mathbb{C}^{p+1}$, we can define an action of $\mathbb{S}^1 \subset \mathbb{C}$ by

$$\begin{aligned} \mathbb{S}^1 \times \mathbb{S}^{2p+1} &\to \mathbb{S}^{2p+1} \\ (\lambda, (z_1, \dots, z_{p+1})) &\mapsto (\lambda z_1, \dots, \lambda z_{p+1}). \end{aligned} \tag{9.9.2}$$

It is easy to see that since $|\lambda| = 1$, the action sends an element of the sphere to another element of the sphere. The quotient of the sphere by this action is, similarly to the real case in which we consider the quotient of the sphere by the antipodal action to get the real projective space, the complex projective space. The following homeomorphisms show it well

$$\mathbb{P}^p(\mathbb{C}) \simeq \left(\mathbb{C}^{p+1} \setminus \{0\}\right)/_{\sim} \simeq \mathbb{S}^{2p+1}/\mathbb{S}^1. \tag{9.9.3}$$

where the relation $\sim$ is defined by

$$(z_1, \dots, z_{p+1}) \sim (z_1', \dots, z_{p+1}') \tag{9.9.4}$$

if and only if there exists $\lambda \in \mathbb{C}^*$ such that

$$z_k' = \lambda z_k \tag{9.9.5}$$

for all k. The second homeomorphism is therefore obtained by restricting to the sphere, and keeping only the action of the complexes of module equal to 1. ☐

Now, recall that our goal is to study the behavior of the volume form of $\mathbb{P}^p(\mathbb{C})$ under the action of the involution coming from the real structure, so we will begin by checking the behaviour of the form volume of $\mathbb{S}^{2p+1}$ under the same involution and then see how this changes through the quotient map.

Proposition 9.10. *The involution on $\mathbb{S}^{2p+1}$ is given by the usual conjugation on $\mathbb{C}^{p+1}$ or equivalently, through the identification of the coordinates, by the following involution of $\mathbb{R}^{2p+2}$:*

$$\begin{aligned} \sigma : \mathbb{R}^{2p+2} &\to \mathbb{R}^{2p+2} \\ (x_1, x_2, \dots, x_{2p+2}) &\mapsto (x_1, -x_2, \dots, x_{2p+1}, -x_{2p+2}). \end{aligned} \tag{9.10.1}$$

Thus, the action of σ is sending the form ω defined in 9.8.4 onto

$$\sigma_*\omega = \sum_{k=1}^{2p+1} (-1)^k(-1)^p x_k dx_1 \wedge \dots \wedge dx_{k-1} \wedge dx_{k+1} \wedge \dots \wedge dx_{2p+2} = (-1)^p\omega. \tag{9.10.2}$$

The same holds for the pullback form $\iota^*\omega$ on the $(2p+1)$-sphere.

To get the volume form on $\mathbb{P}^p(\mathbb{C})$, one way is to consider he *interior product* of the volume form of the sphere with the vector field that generates the action of $\mathbb{S}^1$ seen as a Lie group.

Proposition 9.11. *The vector field generating the action of he Lie group $\mathbb{S}^1$ on the sphere $\mathbb{S}^{2p+1}$ is the vector field X defined as follows*

$$X(x_1, x_2, \dots, x_{2p+2}) = (-x_2, x_1, -x_4, x_3, \dots, -x_{2p+2}, x_{2p+1}). \tag{9.11.1}$$

Proof. That is easy to check from the definition of the action of $\mathbb{S}^1$ on the sphere, which can be written as follows

$$(a+ib)\cdot(x_1, x_2, \dots, x_{2p+2}) = (ax_1 - bx_2, bx_1 + ax_2, \dots, ax_{2p+1} - bx_{2p+2}, bx_{2p+1} + ax_{2p+2}). \tag{9.11.2}$$

Making a converging towards 0 and b towards 1 to get the infinitesimal action, we get the result. ☐

Theorem 9.12. *We get a volume form of $\mathbb{P}^p(\mathbb{C})$ by taking the interior product of the vector field X with the volume form on $\mathbb{S}^{2p+1}$, $\iota_X\omega$. It is a $(2p)$-form defined as*

$$\iota_X\omega(X_1, X_2, \dots, X_{2p}) = \omega(X, X_1, X_2, \dots, X_{2p}). \tag{9.12.1}$$

Furthermore, the action of the involution on this form is the same as on the volume form of $\mathbb{S}^{2p+1}$, that is to say

$$\sigma_*\iota_X\omega = (-1)^p\iota_X\omega. \tag{9.12.2}$$

Proof. The volume form ω is invariant under the action of the vector field X and horizontal to it, therefore the form $\iota_X\omega$ goes through the quotient and defines a volume form of $\mathbb{P}^p(\mathbb{C})$, and the action of σ is exactly the same as the one on the original form ω. To see that, we will prove it in the case $p = 1$, the other being the same. We begin with the form ω on $\mathbb{S}^3 \subset \mathbb{R}^4$:

$$\omega = x_1 dx_2 \wedge dx_3 \wedge dx_4 - x_2 dx_1 \wedge dx_3 \wedge dx_4 + x_3 dx_1 \wedge dx_2 \wedge dx_4 - x_4 dx_1 \wedge dx_2 \wedge dx_3. \tag{9.12.3}$$

Then the interior product with the vector field on $\mathbb{S}^3$ is equal to

$$\begin{aligned} \iota_X\omega =& x_2\left(x_2 dx_3 \wedge dx_4 - x_3 dx_2 \wedge dx_4 + x_4 dx_2 \wedge dx_3\right) \\ &+ x_1\left(x_1 dx_3 \wedge dx_4 - x_3 dx_1 \wedge dx_4 + x_4 dx_1 \wedge dx_3\right) \\ &+ x_4\left(x_1 dx_2 \wedge dx_4 - x_2 dx_1 \wedge dx_4 + x_4 dx_1 \wedge dx_2\right) \\ &+ x_3\left(x_1 dx_2 \wedge dx_3 - x_2 dx_1 \wedge dx_3 + x_3 dx_1 \wedge dx_2\right). \end{aligned} \tag{9.12.4}$$

Now, we can note that every term is a combination of an odd number of factors x_2, dx_2, x_4 and dx_4, which are the ones shifting the sign under the action of the involution σ, thus we have the equality

$$\sigma_* \iota_X \omega = -\iota_X\omega. \tag{9.12.5}$$

The quotient does not alter this identity and therefore the same holds for the volume form on $\mathbb{P}^1(\mathbb{C})$. $\square$

As the volume form of $\mathbb{P}^p(\mathbb{C})$ defines the cohomology class of maximal degree, we get the proposition 9.7 that we wanted. From this same proposition 9.7, we get that if d is even, the left-hand side is equal to:

$$\begin{aligned} \prod_{\substack{0\le m\le 2d,\\ m \text{ even}}} L_{\mathbb{R}}\left(H^m\left(\mathbb{P}^d\left(\mathbb{C}\right)\right), s\right)^{+1} &= \prod_{\substack{p=0\\ p\, even}}^{d} \Gamma_{\mathbb{R}}(s-p)^{+1} \prod_{\substack{p=1\\ p\, odd}}^{d-1} \Gamma_{\mathbb{R}}(s-p+1)^{+1} \\ &= \left(\prod_{\substack{p=0,\\ p\, even}}^{d-2} \Gamma_{\mathbb{R}}(s-p)^2\right) \cdot \Gamma_{\mathbb{R}}(s-d). \end{aligned} \tag{9.12.6}$$

And if d is odd, it is:

$$\begin{aligned} \prod_{\substack{0\le m\le 2d,\\ m \text{ even}}} L_{\mathbb{R}}\left(H^m\left(\mathbb{P}^d\left(\mathbb{C}\right)\right), s\right)^{+1} &= \prod_{\substack{p=0\\ p\, even}}^{d-1} \Gamma_{\mathbb{R}}(s-p)^{+1} \prod_{\substack{p=1\\ p\, odd}}^{d} \Gamma_{\mathbb{R}}(s-p+1)^{+1} \\ &= \prod_{\substack{p=0,\\ p\, even}}^{d} \Gamma_{\mathbb{R}}(s-p)^2. \end{aligned} \tag{9.12.7}$$

The two functions agree in every case.

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