Representations of linear groups: A dynamical point of view

Lecture notes

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Introduction

This course is an introduction to the area of mixing an equidistribution in Lie groups. We will see on a rather simple example how rich can be a mixture of geometric, algebraic and dynamical concepts.

The main result of these lectures is an analog in the hyperbolic case of a simple result in planar geometry: consider the tiling of $\mathbb{R}^2$ by squares of side 1. Then you can count the number of vertices of this tiling inside a disc of radius R:

Proposition 1. *The number of points in $\mathbb{Z}^2$ which are inside the ball $B(0,R)$ is equivalent to the area of this disc.*

Moreover you can state this result for any regular tiling of the plan. The proof is very simple: the area of the disc is approximated by the sum of the areas of the tiles inside. The error term is then along the sphere and can be neglected. We will see that this proof does not generalize to the hyperbolic geometry.

The three lectures are organized in the following way:

- The first one is dedicated to a presentation of the hyperbolic plane, the action of $\mathrm{SL}(2,\mathbb{R})$ by homography and the modular group. The main result of this course is stated at the end.
- The second one is devoted to an important ingredient of the proof: the decay of coefficients for unitary representations of $\mathrm{SL}(2,\mathbb{R})$ and its interpretation as a mixing property.
- The third one includes the proof of the main result and some remarks on its possible generalizations.

All the materials and ideas presented here are rather classical and can be found in the thesis of Margulis. We will see at the end several recent results inspired by these works.

Chapter 1

Hyperbolic geometry and lattices in $\mathrm{SL}(2,\mathbb{R})$

We present here some features of the hyperbolic geometry needed to state the main result.

1.1 Hyperbolic plane and disc

1.1.1 Definitions

The hyperbolic plane is a riemannian manifold over the set

$$\mathbb{H} = \{z \in \mathbb{C} \text{ such that } \mathrm{Im}\ z > 0\}$$

with its natural structure of smooth manifold. And, at each point $z \in \mathbb{H}$, we define a norm on the tangent space, canonically identified with $\mathbb{C}$, by the formula:

$$\text{For each } z \in \mathbb{H} \text{ and } v \in \mathbb{C},\ \|v\|_z = \frac{|v|}{\mathrm{Im} z}\ ;$$

As usual, this data allows us to define the length of a smooth curve $\gamma : [0,1] \to \mathbb{H}$ by the formula:

$$\mathrm{length}(\gamma) = \int_0^1 \|\gamma'(t)\|_{\gamma(t)} dt = \int_0^1 \frac{|\gamma'(t)|}{\mathrm{Im}\ \gamma(t)} dt\ .$$

This formula is invariant by a change of parametrization of γ so it defines the length of the geometrical curve. And the hyperbolic distance d between two points in $\mathbb{H}$ is the minimal length of a smooth curve between the two points.

By construction, at each point, the hyperbolic metric is homothetic to the euclidean one: therefore the notion of angle between two curve is well-defined. In fact, the hyperbolic metric is conform for the natural structure of Riemann surface on $\mathbb{H}$.

The application $z \mapsto \frac{i-z}{i+z}$ maps the space $\mathbb{H}$ to the open disc. This application is holomorphic on $\mathbb{H}$ so the image of the hyperbolic metric still is a conform metric.

We hence get two isometric models of the same geometry. But what are the straight lines, or geodesics, in this geometry ? The following fact gives part of the answer: the vertical line d_0 passing at i is a straight line:

Fact 1. *Let t_1 and t_2 be in $\mathbb{R}_+^*$.*
Then the hyperbolic distance between it_1 and it_2 is $d(it_1, it_2) = |\ln(\frac{t_1}{t_2})|$ and is given by the hyperbolic length of the segment joining this two points.

Proof. Exercise. □

To completely answer the question, we can now use the action by homography of the group $\mathrm{SL}(2,\mathbb{R})$. This action makes the hyperbolic geometry a very rich one.

1.1.2 The action of $\mathrm{SL}(2,\mathbb{R})$

The group $\mathrm{SL}(2,\mathbb{R})$ acts on the space $\mathbb{H}$ by the following:

$$\text{For all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}), \text{ and } z \in \mathbb{H}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \bullet z = \frac{az+b}{cz+d} .$$

For all $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$, we note $\phi_g : z \mapsto \frac{az+b}{cz+d}$ the associated homography of $\mathbb{H}$.

Exercise 1. *Check that it is indeed a group action .*

A rapid computation shows that $\mathrm{Im}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \bullet z\right) = \frac{\mathrm{Im}\, z}{|cz+d|^2}$. We are now able to understand how this action transforms the metric: if $v \in \mathbb{C}$ is a tangent vector at $z \in \mathbb{H}$, its image under $D\phi_g$ is the tangent vector $\phi_g'(z)v = \frac{v}{(cz+d)^2}$ at $\phi_g(z)$. So its hyperbolic norm is:

$$\|\phi_g'(z)v\|_{\phi_g(z)} = \frac{|\phi_g'(z)v|}{\mathrm{Im}\,(\phi_g(z))} = \frac{|v|}{\mathrm{Im}\, z} = \|v\|_z .$$

To state it with words rather than equations: the action of $\mathrm{SL}(2,\mathbb{R})$ by homography on the hyperbolic plane preserves the hyperbolic metric.

Another interesting feature is the 2-transitivity of the action:

Fact 2. *Let z_1 and z_2 be two points of $\mathbb{H}$.*
Then it exists $g \in \mathrm{SL}(2,\mathbb{R})$ such that $g \bullet z_1 = i$ and $g \bullet z_2 = ie^{d(z_1,z_2)}$.

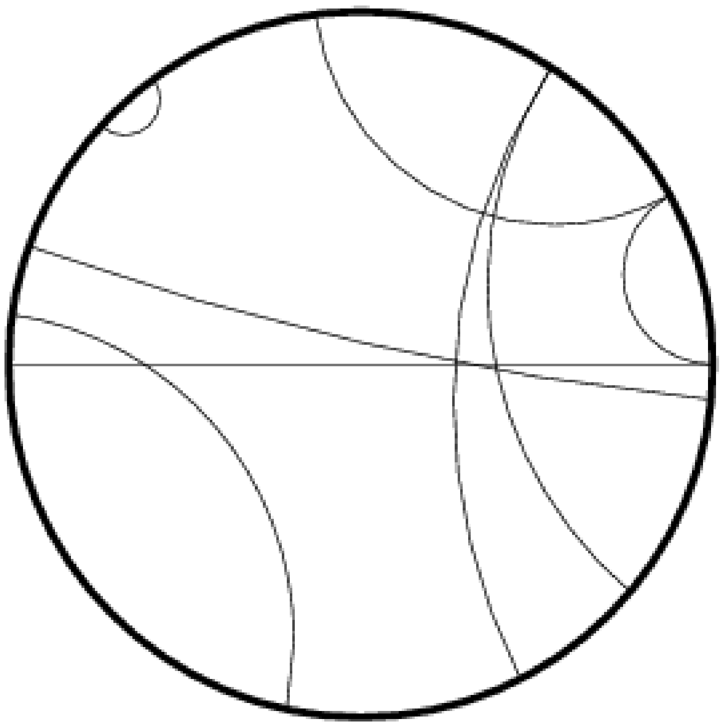
This fact implies two different statements:

- The hyperbolic plane is an homogeneous space under $\mathrm{SL}(2,\mathbb{R})$. Indeed it admits the following description, once checked that the stabilizer of i is the subgroup $\mathrm{SO}(2)$:

  $$\mathbb{H} \simeq \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2) .$$

  Then the whole group $\mathrm{SL}(2,\mathbb{R})$ identifies naturally with the unit tangent bundle $T_1\mathbb{H}$ of $\mathbb{H}$.

- The geodesics for this geometry are the images under homography of the vertical line. They are either vertical lines or half-circle orthogonal to the real axis. In the disc model, they are also circles or lines orthogonal to the unit circle:



1.1.3 Geodesics, horizon and Cartan decomposition

We just described the sets of geodesics. We remark that a geodesic in the hyperbolic plan always meet the set $\mathbb{R} \cup \{\infty\}$ -called frontier at ∞, or horizon- at exactly two points and is fully determined by this intersection. Let $\mathcal{G}$ be the set of oriented geodesic in $\mathbb{H}$. Then the action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{H}$ naturally defines a action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathcal{G}$, and then on this frontier at infinity. This action is exactly the action of $\mathrm{SL}(2,\mathbb{R})$ by homography on the real projective line.

Moreover $\mathrm{SL}(2,\mathbb{R})$ acts on the set $\{(z,d)$ such that $z \in \mathbb{H}$, $d \in \mathcal{G}$ and $z \in d\}$. This action is almost simply transitive: only Id and $-Id$ - which both act as identity - preserve the couple (i, d_0).

This geometric description may in turn contains some algebraic information. Let's give as an example a geometric proof of the Cartan decomposition:

Proposition 2. *Let g be an element of $\mathrm{SL}(2,\mathbb{R})$. Then it exists a unique $t > 1$ and there exist k_1, k_2 in $\mathrm{SO}(2)$ such that:*

$$g = k_1 \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} k_2 \ .$$

Proof. The element g sends the couple (i, d_0) to a couple (z, d). We may suppose that z is another point than i, as, if not, g would be in SO(2) and the Cartan decomposition is trivial. We can decompose the motion: let k be the element of SO(2) which maps z on $it^2 := ie^{d(i,z)}$. Then $k \bullet z = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \bullet i$. So the element $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}^{-1} kg$ fixes the point i. It sends d_0 on a given oriented geodesic d' containing i: there exist k' in SO(2) mapping d' on d_0. Hence we have:

$$\left(k' \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}^{-1} kg \right) \bullet i = i$$
$$\left(k' \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}^{-1} kg \right) \bullet d_0 = d_0$$

As seen before, we now know that

$$g = \pm k^{-1} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} (k')^{-1}$$

So the existence of the decomposition is proven. The uniqueness of t comes from the equality $t = e^{\frac{d(i, g\bullet i)}{2}}$. □

1.1.4 Hyperbolic area, spheres and balls

The hyperbolic metric defines an hyperbolic area on $\mathbb{H}$: for all domain D in $\mathbb{H}$, its area is

$$\text{area}(D) = \int\int_D \frac{dxdy}{y^2} .$$

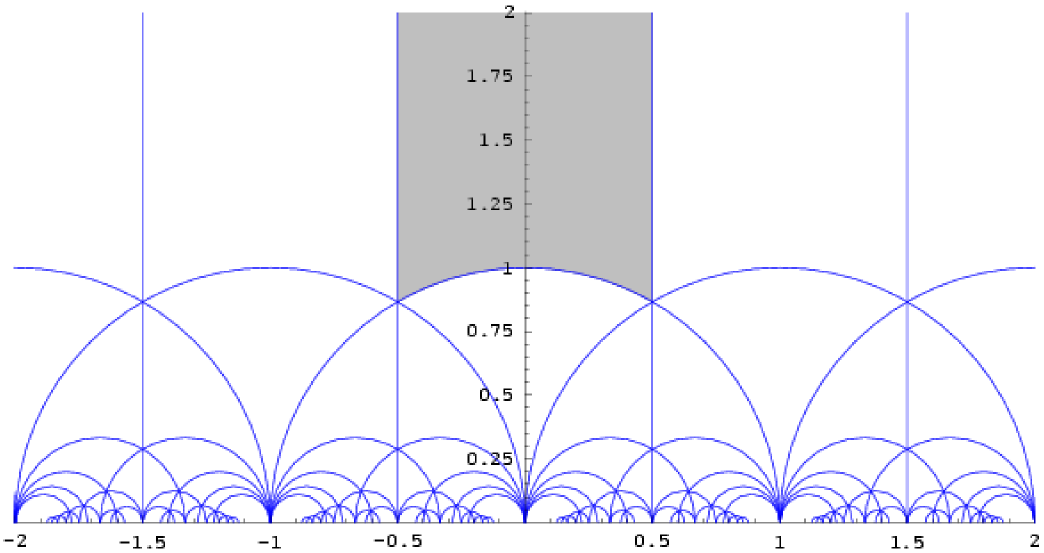
We will encounter the ball of radius R and center i and need its area. First of all, the sphere of center i and radius R is the orbit under SO(2) of the point ie^R. One checks that this is the circle of center $i\cosh(R)$ and radius $\sinh(R)$. So one can compute its area: $area(B(i, R) = 4\pi \sinh^2(\frac{r}{2})$.

This exponential rate of growth of the area of balls shows a maybe surprising feature: the area of a small annulus near the sphere of radius R also grows exponentially so as fast as the area of the whole ball. At this point one can recall that in the euclidean case we need to neglect some border terms. And this is possible because the area of a small annulus near the sphere grows slower than the area of the disc. This difference between hyperbolic and euclidean geometry is crucial here.

1.2 Action of the modular group

1.2.1 The modular group and surface

The modular group is the subgroup $\mathrm{SL}(2,\mathbb{Z})$ of integral matrices in $\mathrm{SL}(2,\mathbb{R})$. It's clearly a discrete subgroup. A set of generators of this group is $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$, $\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ and $-Id$. Hence its actions on $\mathbb{H}$ is generated by the two transformations $z \mapsto -z^{-1}$ and $z \mapsto z+1$. So this action admits a fundamental domain:



This domain is a hyperbolic triangle with angles 0, $\frac{\pi}{3}$ and $\frac{\pi}{3}$, so its area is $\frac{\pi}{3}$.

We call lattice in $\mathrm{SL}(2,\mathbb{R})$ such a group: a discrete subgroup Γ of $\mathrm{SL}(2,\mathbb{R})$ such that the fundamental domain is of finite hyperbolic area. The second hypothesis is equivalent here to the finiteness of the volume of $\Gamma\backslash G$ for the Haar measure on G. If the fundamental domain is compact, the lattice is said uniform.

The quotient space

$$\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H} \simeq \mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$$

is called the modular surface and is the modular space of elliptic curves. It has a structure of Riemann surface and an hyperbolic metric with two conical singularities of order 2 and 3. And, as before, the space $\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R})$ identifies with the unit tangent bundle.

1.2.2 Counting points in balls

We will prove during this three lecture the following theorem:

Theorem 1. *The number of points in* $\mathrm{SL}(2,\mathbb{Z})\bullet i$ *which are inside the ball* $B(i,R)$ *grows as* $\frac{3}{2\pi}\mathrm{area}(B(i,R))$.

We already discussed the fact that we won't be able to use the same strategy as in the euclidean case because we can't neglect the border terms. So we will have to study precisely how the sphere of radius R crosses the tiling of the hyperbolic plane, or equivalently how this sphere projects in the modular surface.

The reader may find more on links between lattices in $\mathrm{SL}(2,\mathbb{R})$ and tilings of the hyperbolic plane in the book [1].

Chapter 2

Unitary representations of $\mathrm{SL}(2,\mathbb{R})$

This chapter is devoted to the study the so-called "decay of coefficient" of unitary representations of $\mathrm{SL}(2,\mathbb{R})$. We will prove Howe-Moore theorem for $\mathrm{SL}(2,\mathbb{R})$ using Mautner phenomenon. Next, we mention how Howe-Moore theorem may be understood as a result on actions of $\mathrm{SL}(2,\mathbb{R})$ on hyperbolic surfaces, namely stating the mixing property.

2.1 Unitary representations

2.1.1 Definitions

In this section, we consider a locally compact group G. Let's define the notion of unitary representation:

Definition 1. *An unitary representation of G is the following data:*

- *a Hilbert space $\mathcal{H}$ called the space of representation,*
- *a morphism ρ from G to the group of unitary transformations of $\mathcal{H}$, such that for all $v \in \mathcal{H}$, the application: $\begin{array}{ccc} G & \to & \mathcal{H} \\ g & \mapsto & \rho(g)v \end{array}$ is continuous.*

To such a representation is associated a family of functions from G to $\mathbb{C}$ called the coefficients of the representation:

Definition 2. *For all v, w in $\mathcal{H}$, a coefficient of the representation is the function:*

$$c_{v,w} \ : \ \begin{array}{ccc} G & \to & \mathcal{H} \\ g & \mapsto & < \rho(g)v, w > \end{array}.$$

We remark that when $\mathcal{H}$ is finite dimensional, the matrix entries of the representation written in a given base are indeed coefficients of the representation.

Let us give a few examples. First of all, the trivial representation in any Hilbert space is clearly a unitary representation. In this case, all the coefficients are constant functions.

More interesting is the following: we suppose that G acts (on right) continuously on a locally compact space X preserving a borelian measure μ. Then G acts continuously on the space $L^2(X, \mu)$ by the following formula:

$$\text{For } g \in G,\ \phi \in L^2(X, \mu) \text{ and almost all } x \in X\ : (g \bullet \phi)(x) = \phi(x.g)\ .$$

We can check that this action is in fact a unitary representation of G in the Hilbert space $L^2(X, \mu)$: the continuity hypothesis is fulfilled because the action itself is continuous. And, as G preserves the measure μ, we have:

$$\|g \bullet \phi\|_2^2 = \int_X \phi(x.g)\bar{\phi}(x.g)d\mu(x) = \int_X \phi(x)\bar{\phi}(x)d\mu(x) = \|\phi\|_2^2$$

which proves that the representation is unitary. In this case the coefficient at ϕ, ψ is the function $g \mapsto \int_X \phi(xg)\bar{\psi}(x)d\mu(x)$.

Let's recall from the precedent lecture that we are interested in understanding some phenomena which take place in the modular surface $\mathrm{SL}(2, \mathbb{Z})\backslash\mathbb{H}$ or its unit tangent bundle $\mathrm{SL}(2, \mathbb{Z})\backslash\mathrm{SL}(2, \mathbb{R})$. But this last space is a locally compact space on which $\mathrm{SL}(2, \mathbb{R})$ acts continuously preserving the Haar measure. So the precedent remark gives us a unitary representation of $\mathrm{SL}(2, \mathbb{R})$. We will need in the third lecture a result stating that the non-constant coefficients of this representation tend to 0 when g goes to ∞ (it means that g leaves all compact subsets of $\mathrm{SL}(2, \mathbb{R})$). This is the phenomenon called decay of coefficient and is given by Howe-Moore theorem for $\mathrm{SL}(2, \mathbb{R})$:

Theorem 2 (Howe-Moore)**.** *Let $(\rho, \mathcal{H})$ be a unitary representation of* $\mathrm{SL}(2, \mathbb{R})$ *without any globally fixed point in $\mathcal{H}$.*

Then for all v, w in $\mathcal{H}$, the coefficient $c_{v,w}(g)$ goes to 0 as g goes to ∞ in $\mathrm{SL}(2, \mathbb{R})$.

2.1.2 Invariance by diagonal or unipotent elements

We will present in this section the Mautner phenomenon: for a vector v of the space of an unitary representation of $\mathrm{SL}(2, \mathbb{R})$ to be invariant, it is enough to be fixed by either a single (non trivial) diagonal element or a single (non trivial) unipotent element. Let's give some subgroup notations:

$$\begin{aligned} A &= \left\{a(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \text{ for } t > 0\right\} \\ U^+ &= \left\{u^+(s) = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \text{ for } s \in \mathbb{R}\right\} \\ U^- &= \left\{u^-(s) = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \text{ for } s \in \mathbb{R}\right\} \end{aligned}$$

One can easily check that $\mathrm{SL}(2, \mathbb{R})$ is generated by this three subgroups. We will prove the following proposition:

Proposition 3. *Let $(\rho, \mathcal{H})$ be a unitary representation of* $\mathrm{SL}(2,\mathbb{R})$ *and v be an element of $\mathcal{H}$.*

If it exists some $t > 0$ such that $\rho(a(t))v = v$ or some $s \in R$ such that $\rho(u^+(s))v = v$ then v is fixed by the whole group $\mathrm{SL}(2,\mathbb{R})$.

Proof. We will prove in fact the following equivalence, for $t > 1$, s and s' some real numbers:

$$\rho(a(t))v = v \Leftrightarrow \rho(u^+(s))v = v \Leftrightarrow \rho(u^-(s'))v = v \ .$$

By symmetry it is enough to show the first equivalence for $t > 1$. The following lemma is the key point:

Lemma 1. *Let v be in $\mathcal{H}$ and* $\mathrm{Stab}(v)$ *its stabilizer in* $\mathrm{SL}(2,\mathbb{R})$. *Consider some element g of* $\mathrm{SL}(2,\mathbb{R})$. *If there exist three sequences (g_n) in* $\mathrm{SL}(2,\mathbb{R})$ *and (h_n), $(h_n^{'})$ in* $\mathrm{Stab}(v)$ *such that:*

- *g_n goes to g with n,*
- *$h_n g_n h_n^{'}$ goes to Id,*

then g belongs to $\mathrm{Stab}(v)$.

Proof. Let's compute the product $< \rho(g).v, v >$: by the assumptions it is equal to the limit in n of $< \rho(g_n).(\rho(h_n^{'}).v), \rho(h_n^{-1}).v >$. As the representation is unitary, the last product is equal to $< \rho(h_n g_n h_n^{'}).v, v >$, hence goes to $\|v\|^2$. So we have proven that $< \rho(g).v, v >= \|v\|^2$, which implies that $\rho(g).v = v$. The lemma is proven. □

We can now prove the equivalence:

First, if v is fixed by $a(t)$, we want to prove that it is also fixed by $u^+(s)$. Apply the precedent lemma with $g = g_n = u^+(s)$, $h_n^{'} = h_n^{-1} = a(t)$.

Second, if v is fixed by $u^+(s)$, we want to prove that it is also fixed by $a(t)$. The lemma can another time be applied with the following data: take k_n and l_n two sequences of integers going to ∞ such that $|\frac{k_n}{l_n} + t| \leq \frac{1}{l_n^2}$. Then define the real sequence (α_n) by: $\alpha_n = \frac{1-t}{k_n s}$. Now we can consider:

$$\begin{aligned} g_n &= \begin{pmatrix} t & 0 \\ \alpha_n & t^{-1} \end{pmatrix} \\ h_n &= u^+(k_n s) = (u^+(s))^{k_n} \\ h_n^{'} &= u^+(l_n s) = (u^+(s))^{l_n} \end{aligned}$$

and check that $g_n \to a(t)$, h_n and $h_n^{'}$ are in Stab(v) as power of $u^+(s)$ and $h_n g_n h_n^{'} \to Id$.

This proves the given equivalence and finishes the proof of the proposition. □

2.1.3 Decay of coefficients

We will now prove Howe-Moore theorem by contradiction: suppose there is some coefficient which doesn't go to 0. Then by extracting convergent sequences, you can construct a vector which is invariant under the action of $u^+(1)$. The Mautner phenomenon hence implies that this vector is globally invariant which is the contradiction.

Proof. We assume the conclusion to be false. Hence we get two vectors v and w in $\mathcal{H}$, and a sequence g_n in $\mathrm{SL}(2,\mathbb{R})$ going to ∞ such that the coefficient $c_{v,w}(g_n) =< \rho(g_n)v, w >$ is bounded from 0. So, up to an extraction, we may assume that it goes to some $\alpha \neq 0$.

Using the Cartan decomposition stated in the precedent chapter, we write: $g_n = k_n^1 a(t_n) k_n^2$, where the sequences (k_n^1) and (k_n^2) are in $\mathrm{SO}(2)$ and t_n is greater than 1. As g_n goes to ∞ and $\mathrm{SO}(2)$ is compact, the sequence (t_n) goes to $+\infty$ with n. Moreover, we may assume - up to an extraction and a change of the vectors v, w - that (k_n^1) and (k_n^2) both go to Id in $\mathrm{SO}(2)$.

We know that the balls of the Hilbert space $\mathcal{H}$ are weakly compact: up to another extraction, the sequence $\rho(a(t_n))v$ has a weak limit in the ball of radius $\|v\|$, noted v_0. This means that for any vector $v' \in \mathcal{H}$, the product $< v_0, v' >$ is the limit of $< \rho(a(t_n))v, v' >$. We want to check that v_0 is the requested vector: it must be different from 0 and invariant by $u^+(1)$:

- we have by definition $< v_0, w >= \lim < \rho(a(t_n)v, w >$. As $\rho(a(t_n))$ is unitary and the sequences (k_n^1), (k_n^2) go to Id, we may write

$$< v_0, w >= \lim < \rho(a(t_n)(\rho(k_n^2)v), \rho(k_n^1)^{-1} w > \ .$$

  Now this last product is exactly $< \rho(g_n)v, w >$ hence goes to α. We have just proven that $< v_0, w >= \alpha \neq 0$, so v_0 is different from 0.

- Consider $u^+(1) \in U^+$. First of all, we check that

$$a(t_n)^{-1} u^+(1) a(t_n) = u^+(t_n^{-2}) \xrightarrow{n\to\infty} 0 \ .$$

  Then we may once again use the definition of v_0 as a weak limit to write:

$$\begin{aligned} < \rho(u^+(1))v_0, v_0 > &= \lim < \rho(u^+(1)a(t_n))v, v_0 > \\ &= \lim < \rho(a(t_n)\left(\rho(a(t_n)^{-1}u^+(1)a(t_n))\right) v, v_0 > \\ &= \lim < \rho(a(t_n)v, v_0 > \\ &= < v_0, v_0 > \end{aligned}$$

  Hence $\rho(u^+(1))v_0 = v_0$.

That means that v_0 is a non-null vector invariant under $u^+(1)$. The Mautner phenomenon now implies that v_0 is fixed by the whole group $\mathrm{SL}(2,\mathbb{R})$ which is a contradiction with the hypothesis.

So we have proven Howe-Moore theorem. □

This theorem is valid for any reductive algebraic group. Moreover it is closely related to property (T) of Kazhdan which gives some rate of decay for Lie groups of higher rank (see [2, Part 3]). Getting a rate of decay for some representations of SL(2) is more involved. It requires some result toward Ramanujan conjecture for SL(2), for example using the trace formula of Selberg. We will not go farther in that direction during this lecture.

2.2 Mixing

This section is devoted to a dynamical interpretation of Howe-Moore theorem. We will briefly expose the mixing property for a single transformation of a space and then restate Howe-Moore theorem.

2.2.1 The mixing property

Consider (X, m) a probability space and T a transformation of X which preserves m.

Definition 3. *The transformation T is said to be mixing if for all open subsets A and B of X we have the following:*

$$\frac{m(T^{-n}(B) \cap A)}{m(A)} \xrightarrow{n \to \infty} m(B) \ .$$

Remark that we can restate this in a probabilistic language saying that the event "being inside B after n steps" becomes more and more independent of "being inside A at the beginning". Another way to say it is that the information on your initial state becomes irrelevant to predict your future. This property is a way to describe a system as chaotic. An example of such a chaotic system is the following: (X, m) is the circle with its standard Lebesgue measure and T is the transformation which multiplies the angles by 2.

The following remark links this property to the decay of coefficient: using an integral notation and the density of simple functions, we see that the mixing property can be written

$$\text{For all } \phi \text{ and } \psi \in L^2(X, m), \text{ we have } \int_X \phi(x)\bar{\psi}(T^n x) dm(x) \xrightarrow{n \to \infty} \left(\int_X \phi\right)\left(\int_X \psi\right) \ ,$$

or, by projection on the subspace $L^2_0(X, m)$ of function of null mean:

$$\text{For all } \phi \text{ and } \psi \in L^2_0(X, m), \text{ we have } \int_X \phi(x)\bar{\psi}(T^n x) dm(x) \xrightarrow{n \to \infty} 0 \ .$$

2.2.2 Mixing of the actions of $\mathrm{SL}(2, \mathbb{R})$

Now we can reinterpret what we have seen on the actions of $\mathrm{SL}(2, \mathbb{R})$ in a new vocabulary. Indeed, we have seen that any lattice Γ in $\mathrm{SL}(2, \mathbb{R})$ gives a space $X = \Gamma \backslash \mathrm{SL}(2, \mathbb{R})$ of finite volume. Up to a normalization it is a probability space. And $\mathrm{SL}(2,\mathbb{R})$ acts transitively on X preserving the measure.

We then get a unitary representation of $\mathrm{SL}(2,\mathbb{R})$ in the space $L^2(X)$. The globally fixed points are the constant functions by transitivity of the action of $\mathrm{SL}(2,\mathbb{R})$ on X. So we can apply Howe-Moore theorem to the representation of $\mathrm{SL}(2,\mathbb{R})$ in the space $L^2_0(X)$ of functions of null mean. Using the expression of the coefficients in this case we get the following property:

$$\text{For all } \phi \text{ and } \psi \in L^2_0(X,m), \text{ we have } \int_X \phi(x)\bar{\psi}(gx)dm(x) \xrightarrow{g\to\infty} 0 \ .$$

It is exactly the mixing property for the action of $\mathrm{SL}(2,\mathbb{R})$ on X. Hence a corollary of Howe-Moore theorem is the following:

Corollary 1. *The action of* $\mathrm{SL}(2,\mathbb{R})$ *on either the unit tangent bundle of any hyperbolic surface of finite area or the surface itself is mixing.*

Chapter 3

Counting, Mixing and Equidistribution

Now that we have explained the needed tools, we may go on with the proof of the main result:

Theorem 3. *The number of points in* $\mathrm{SL}(2,\mathbb{Z}) \bullet i$ *which are inside the ball* $B(i,R)$ *grows as* $\frac{3}{2\pi}\mathrm{area}(B(i,R))$.

The idea is the following: the hyperbolic plan is tiled by translate of the fundamental domain, and to each point of the orbit is associated the tile it belongs to. We want to prove that the sum of the areas of tiles associated to points inside the hyperbolic disk of radius R is equivalent to the area of the ball. So we want to evaluate the error term. The mixing property then allow us to prove that the error term goes to 0 because the sphere of radius R meets "randomly" the tiles.

This chapter is directly inspired from the introduction of the article of Eskin and MacMullen [5, Section 2]

3.1 Equidistribution of spheres

We prove here the first point: the projections of the spheres in the modular surface tend to be uniformly distributed. To state this properly, recall that the group $SO(2)$ is the group of the circle and we have a uniform probability measure on it, noted λ. Then by projecting the sphere in the modular surface we get a probability measure on it. We affirm that as the radius of the sphere goes to ∞, this probability measure goes to the normalized area on the modular surface. We note p the projection from the plan $\mathbb{H}$ to $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$. Moreover, we note m the area measure on $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$ locally proportional to the area measure and of volume $\frac{\pi}{3}$. Last we note $\tilde{m}$ the measure on its unit tangent bundle $\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R})$ which is the product of the measure m in the base and the uniform measure in the fibers which are isomorph to $\mathrm{SO}(2)$.

Theorem 4. *Let* ϕ *be any continuous and compactly supported function on* $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$.

Then we have the following limit:

$$\int_{\mathrm{SO}(2)} \phi \circ p(ka(t) \bullet i) d\lambda(k) \xrightarrow{t\to\infty} \frac{3}{\pi} \int_{\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}} \phi dm$$

Remark that the set $\mathrm{SO}(2)a(t) \bullet i$ corresponds to the sphere of radius $2\ln(t)$ in $\mathbb{H}$.

Proof. We will use the mixing property studied in the previous chapter. We have to smoothen a little bit the projection of the sphere, so that the projection of the measure won't be singular. And then we may apply the mixing property. Fix $\varepsilon > 0$.

Remark that if $z = g \bullet i$ is a point in $\mathbb{H}$ with $d(z,i) < a$, then $z' = ga(t) \bullet i$ is at distance less than a from the sphere of radius $2\ln(t)$. But ϕ is compactly supported, so it is uniformly continuous. So we can find a a such that if $d(x,y) < a$, we have $|\phi(x) - \phi(y)| < \varepsilon$. And let $V = \cup_{1\leq t\leq e^{\frac{a}{2}}} \mathrm{SO}(2)a(t)\mathrm{SO}(2)$ be the set of elements in $\mathrm{SL}(2,\mathbb{R})$ which map i inside the ball of radius a and center i.

Now let $f = \frac{\chi_V}{\tilde{m}(V)}$ be the normalized characteristic function of the set V in $\mathrm{SL}(2,\mathbb{R})$. Then we have the following approximation of the integral we are interested in:

$$\left| \int_{\mathrm{SL}(2,\mathbb{R})} f(v)\phi \circ p\,(va(t) \bullet i)\, dm(v) - \int_{\mathrm{SO}(2)} \phi \circ p\,(ka(t) \bullet i)\, d\lambda(k) \right| \leq \varepsilon\ .$$

We define a function F on $\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R})$ by the following formula $F(\mathrm{SL}(2,\mathbb{Z})v) = \sum_{\gamma\in\mathrm{SL}(2,\mathbb{Z})} f(\gamma v)$. We then have by definition:

$$\int_{\mathrm{SL}(2,\mathbb{R})} f(v)\phi \circ p\,(va(t) \bullet i)\, dm(v) = \int_{\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R})} F(x)\phi\,(xa(t) \bullet i)\, dm(x)\ .$$

We can pull back ϕ as a function $\tilde{\phi}$ on $\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R})$ constant in the fibers. The last integral can be expressed in terms of $\tilde{\phi}$:

$$\int_{\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R})} F(x)\phi\,(xa(t) \bullet i)\, dm(x) = \int_{\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R})} F(x)\tilde{\phi}(xa(t))dm(x)\ .$$

Eventually the mixing property concludes the proof:

$$\begin{aligned} \int_{\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R})} F(x)\tilde{\phi}(xa(t))dm(x) &\xrightarrow{t\to\infty} \frac{1}{m(\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R}))} \int_{\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R})} \tilde{\phi} dm \\ &= \frac{3}{\pi} \int_{\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}} \phi dm\ . \end{aligned}$$

Let's summarize this calculus: for all real t, the integral $\int_{\mathrm{SO}(2)} \phi \circ p(ka(t))d\lambda(k)$ is ε-approximated by an integral which goes to $\frac{3}{\pi}\int_{\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}} \phi dm$ as t goes to ∞. As this doesn't depends on ε, that proves the desired limit. □

3.2 Counting

We are now able to conclude the proof of the main statement. Note $N(R)$ the cardinal of the set $(\mathrm{SL}(2,\mathbb{R})\bullet i)\cap B(i,R)$. We want to prove that it is equivalent as R goes to ∞ to $\frac{3}{2\pi}\mathrm{area}(B(i,R))$.

Proof. Fix $\varepsilon > 0$ and let α be a bump function of integral 1 on the hyperbolic plane in a ε-neighborhood of the point i. And $\tilde{\alpha}$ its pushforward on the modular surface:

$$\tilde{\alpha}(\mathrm{SL}(2,\mathbb{Z})x) = \sum_{\gamma\in\mathrm{SL}(2,\mathbb{Z})} \alpha(\gamma x)\ .$$

Then $\tilde{\alpha}$ viewed as a periodic function on the plane is the sum of the translates of 2α around each point of the orbit $\mathrm{SL}(2,\mathbb{Z})\bullet i$ (the 2 comes from the stabilizer of i inside $\mathrm{SL}(2,\mathbb{Z})$). So, we have:

$$2N(R-\varepsilon) \le \int_{B(i,R)} \tilde{\alpha}dm \le 2N(R+\varepsilon)\ .$$

We can now use the polar coordinates to transform the integral in

$$\int_0^R \int_{\mathrm{SO}(2)} \tilde{\alpha}(ka(e^{\frac{t}{2}})\bullet i)d\lambda(k)2\pi sinh(t)dt\ .$$

As the inner integral is equal to

$$\int_{\mathrm{SO}(2)} \alpha\circ p(ka(e^{\frac{t}{2}})\bullet i)d\lambda(k)\ ,$$

the mixing property shows it converges to $\frac{1}{m(\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H})}\int_{\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}}\alpha$ which in turn is equal to $\frac{3}{\pi}$. Hence, using a Cesaro limit theorem, the whole integral is equivalent to $\frac{3}{\pi}\mathrm{area}(B(i,R))$.

That proves the counting statement. □

The same proof is still valid for another initial points. The only difference is a factor 2 as another point will have a trivial stabilizer inside $\mathrm{SL}(2,\mathbb{Z})$. So we get the following result:

Theorem 5. *Let z be a point in the hyperbolic plane which does not belong to the orbit $\mathrm{SL}(2,\mathbb{Z})\bullet i$. Then we have the following equivalence:*

$$\mathrm{Card}\left((\mathrm{SL}(2,\mathbb{Z})\bullet z)\cap B(i,R)\right) \sim \frac{3}{\pi}\mathrm{area}(B(i,R)) \text{ as } R \text{ goes to } \infty.$$

3.3 Going further

The first reference to check in order to study generalizations of these ideas certainly is the paper of Eskin and McMullen itself [5]. They consider the action of lattices in Lie groups on symmetric spaces and get the same type of results. The p-adic and S-arithmetic analog of this is due to Benoist and Oh and may be found in their recent paper [3].

Some other generalizations take place in the setting of adelic groups, with interesting arithmetic applications. The paper of Clozel, Oh and Ullmo [4] shows the benefits of using this setting. Some refinements are found in the paper of Gorodnik-Maucourant-Oh [6] and Guilloux [7].

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