NOTE ON A THEOREM OF BONATTI AND GOMEZ-MONT

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I give another proof of the theorem proved by Bonatti and Gomez-Mont in [BGM01]. This result describes the statistical behaviour of leaves in a flat $\mathbf{P}^1(\mathbf{C})$-bundle over a hyperbolic surface. The idea is to rely on mixing of geodesic flow rather than on rigidity of the unipotent flow. The initial hope was to give an exponential estimate for the convergence. At this point I do not achieve this goal.

1. Introduction and notations

1.1. Suspension of a complex projective structure. Let S be a hyperbolic surface, and Γ a torsion-free lattice in $\mathrm{PSL}(2,\mathbf{R})$ such that $S \simeq \Gamma\backslash\mathbb{H}$. The hyperbolic space $\mathbf{H}$ is the universal cover of S and its unit tangent bundle $U(\mathbf{H})$ is identified to $\mathrm{PSL}(2,\mathbf{R})$. The unit tangent bundle of S, denoted $U(S)$, is thus identified to $\Gamma\backslash\mathrm{PSL}(2,\mathbf{R})$. Both the surface S and its unit tangent bundle carry a natural measure: the hyperbolic area $\mathcal{A}$ on S and the Liouville measure $\mathcal{L}$ on $U(S)$. The latter may be either seen as the extension of $\mathcal{A}$ by the rotation-invariant probability measure in the fibers or as the quotient of Haar measure of $\mathrm{PSL}(2,\mathbf{R})$ under the previous identification of $U(S)$ and $\Gamma\backslash\mathrm{PSL}(2,\mathbf{R})$).

Consider a representation ρ of Γ in $\mathrm{SL}(2,\mathbf{C})$ – e.g. the monodromy of a complex projective structure on S. One constructs its suspension M_ρ: it is a flat $\mathbf{P}^1(\mathbf{C})$-bundle over S constructed as the quotient of $\mathbf{H}\times\mathbf{P}^1(\mathbf{C})$ by the relation $(x,z)\sim(\gamma(x),\rho(\gamma)(z))$ for any $\gamma\in\Gamma$. This space is foliated by quotients of $\mathbf{H}$, the fibers and the foliation being transverse. One constructs along the same path the $\mathbf{P}^1(\mathbf{C})$-fiber space $U(M_\rho)$ over $U(S)$: it is the quotient of $U(\mathbf{H})\times\mathbf{P}^1(\mathbf{C})$ by the relation $(v,z)\sim(\gamma v,\rho(\gamma)(z))$ for any $\gamma\in\Gamma$. Denote by Γ_ρ the subgroup $\{(\gamma,\rho(\gamma)$ for $\gamma\in\Gamma\}$ of $\mathrm{PSL}(2,\mathbf{R})\times\mathrm{PSL}(2,\mathbf{C})$.

We will always denote by π the projections either from $U(S)$ to S or from $U(M_\rho)$ to M_ρ.

1.2. Geodesic flows, Lyapunov exponents and SRB measure. The geodesic flow on $U(S)$ is defined by $X_t(\Gamma x)=\Gamma x a_t$ on $U(S)$, where $a_t=\begin{pmatrix} e^{\frac{t}{2}} & 0\\ 0 & e^{-\frac{t}{2}}\end{pmatrix}$. The Liouville measure $\mathcal{L}$ is invariant and ergodic under the geodesic flow. One then defines the lifted geodesic flow on $U(M_\rho)$: it is geometrically defined by following the geodesic flow along the leaves of the foliation. Algebraically, it is the flow $Y_t^\rho(\Gamma_\rho(x,z))=\Gamma_\rho(xa_t,z)$. It is a multiplicative cocycle over the geodesic flow.

We will always assume that there is no probability measure on $\mathbf{P}^1(\mathbf{C})$ invariant by $\rho(\Gamma)$. This is a very mild assumption, see [BGMV03, Thm 5]. It has a crucial consequence: the multiplicative cocycle Y has a positive Lyapunov exponent λ_ρ [BGMV03, Theorem 4]. It implies that, for almost every direction $v\in U(S)$ and $z\in\mathbf{P}^1(\mathbf{C})$, the projection $z_t(v)$ of $Y_t^\rho(v,z)$ in $\mathbf{P}^1(\mathbf{C})$ converges to a point, denoted $z^+(v)$. This point in $\mathbf{P}^1(\mathbf{C})$ is the eigenline corresponding to the higher Lyapunov exponent in the direction of v. Moreover this convergence is exponentially fast, with exponent being the Lyapunov exponent λ_ρ. By definition, $z^+(v)$ depends only on the positive endpoint in $\partial\mathbf{H}$ of the geodesic directed by v. Define the measurable function $\sigma^+ : U(S) \to U(M_\rho)$ by $\sigma^+(v) = (v, z^+(v))$.

Under the above assumption on ρ, Bonatti, Gomez-Mont and Viana [BGMV03, Prop 0.2 and 3.6] prove that there is a unique SRB measure for Y^ρ with total basin: this measure μ^+ is defined as the pushforward of the Liouville measure $\mathcal{L}$ by σ^+. We denote ν^+ the projection of μ^+ on M_ρ.

1.3. **Equidistribution of spheres.** Let p be a point in M_ρ. Its fiber $U(p)$ in $U(M_\rho)$ is the set of unit vectors at p tangent to the leave through p. The image of $U(p)$ under the lifted geodesic flow Y_t is the set of normal unit vectors pointing outside the sphere of radius t in the leave through p. Let s_t^p be the uniform probability measure on this sphere at time t, and b_t^p be the probability measure on the ball of radius t proportional to the hyperbolic area.

We give another proof that – under the above mentioned assumption on ρ – for any point p in M_ρ, these long spheres and big balls become equidistributed along ν^+:

Theorem 1 ([BGM01]). *Let $S = \Gamma\backslash\mathbf{H}$ be an hyperbolic surface of area $\mathcal{A}(S)$ and $\rho : \Gamma \to \mathrm{PSL}(2,\mathbf{C})$ a representation. Suppose that $\mathbf{P}^1(\mathbf{C})$ does not carry a $\rho(\Gamma)$-invariant probability measure. Then for any point $p \in M_\rho$, the probability measures s_t^p and b_t^p converge to the probability measure $\frac{1}{\mathcal{A}(S)}\nu^+$.*

The proof we propose is an application of the mixing of the geodesic flow on $U(S)$ following the lines of the celebrated paper of Eskin and McMullen [EM93]. Recall that the mixing property is an application of the theory of unitary representations of $\mathrm{PSL}(2,\mathbf{R})$ – namely Howe-Moore theorem. The use of this method in our case does not really require additional work aside from the contracting property of the multiplicative cocycle already mentioned.

It would have been much more interesting to get an *effective* equidistribution using exponential mixing. However our situation seems at first glance to be outside of the usual hypothesis needed to get exponential mixing. In a word, the vectors on which we use the mixing does not at all fulfill usual K-finite hypothesis.

2. Proof of equidistribution

2.1. **Mixing and wavefront lemma.** This is now a standard fact that the geodesic flow is mixing. We stick to the reference [EM93] by refering to their Theorem 1.1. Denote $X = \Gamma\backslash\mathrm{PSL}(2,\mathbf{R})$ on dx its Liouville measure. Let $\mathcal{L}(X)$ be the total mass of X or in other terms the area of S.

Theorem 2. *For any α, β in $\mathrm{L}^2(X)$, we have:*

$$\int_X \alpha(xa_t)\beta(x)dx \xrightarrow{t\to\infty} \frac{\int_X \alpha \int_X \beta}{\mathcal{L}(X)}.$$

In their paper, Eskin and McMullen wanted to prove the equidistribution of long spheres in S (Theorem 1.2). To achieve that one would like to take in the previous theorem β to be a test function and α to be the characteristic function of the unit tangent sphere at a point. Of course the latter is zero as a square integrable function, so you really want to approximate the unit tangent sphere by a small open set. But you should ensure that the evolution of this small open set along the geodesic flow remains near the sphere of radius t. This is the “wavefront lemma” which may be proven (for $\mathrm{PSL}(2,\mathbf{R})$) either by hyperbolic geometry or using Iwasawa decomposition. This is Theorem 3.1 in [EM93]. To state it, let us add a few more notations.

We have choosen a starting point $p = (x, z)$ in M_ρ. Let K be the stabilizer in $\mathrm{PSL}(2,\mathbf{R})$ of x, such that the set of unit tangent vectors to x identifies to K. The sphere of radius t in $\mathbf{H}$ is then Ka_t. Let P be the set of upper triangular matrix in $\mathrm{PSL}(2,\mathbf{R})$. Recall that the Iwasawa decomposition writes: $\mathrm{PSL}(2,\mathbf{R}) = KP$. The wavefront lemma states:

Lemma 3. *For any open neighborhood U of the identity in $\mathrm{PSL}(2,\mathbf{R})$, there exists an open set $V \subset P$ such that*

$$KVa_t \subset Ka_tU$$

for all $t > 0$.

These two results are the key point in the proof of equidistribution of long spheres. We will follow exactly the proof of [EM93] once analyzed the dynamic in the fibers.

2.2. **Contraction in the fibers.** Fix a point $p = (x, z)$ in M_ρ and let $U(p)$ be its fiber in $U(M_\rho)$. As in the previous paragraph, $U(p)$ identifies to (K, z). It then inherits a K-invariant probability measure. Then we have:

Proposition 4. *For almost any $k \in K$, the points $\Gamma_\rho(ka_t, z)$ converge to $\sigma^+(ka_t) = \Gamma_\rho(ka_t, z^+(k))$ with t.*

This proposition is a (very weak) consequence of the existence of the positive Lyapunov exponent: for almost any direction k, the action on the fiber of Y_t becomes close to a violent south-north dynamic on $\mathbf{P}^1(\mathbf{C})$. It is exponentially contracting toward $z^+(k)$ all of $\mathbf{P}^1(\mathbf{C})$ but a point $z^-(k)$. And the set of k such that $z = z^-(k)$ is negligible (w.r.t. the Haar measure on K) because the basin of μ^+ has total mass [BGMV03, Prop. 3.6].

2.3. **Equidistribution.** Let β be a continuous function with compact support on $U(M_\rho)$. We prove in this paragraph the following proposition:

Proposition 5.

$$\int_K \beta(ka_t, z)dk \xrightarrow{t\to+\infty} \int \beta d\mu^+.$$

The announced theorem 1 is an easy consequence of this proposition: let f be a continuous function on M_ρ and take β its lift to a continuous function on $U(M_\rho)$ constant in the fibers. Then we have, by the previous proposition:

$$\begin{aligned}\int_{U(M_\rho)} f ds_t^p &= \int_K \beta(ka_t, z)dk \\ &\xrightarrow{t\to+\infty} \int \beta d\mu^+ \\ &= \int f d\nu^+\end{aligned}$$

An additional integration gives the equidistribution of balls. Let us now show the proposition.

Proof. Fix $\epsilon > 0$. The function β has compact support, hence is uniformly continuous.

First of all, using the contraction in the fibers (proposition 4) and uniform continuity, for t big enough, we have:

$$\left| \int_K \beta(ka_t, z)dk - \int_K \beta(\sigma^+(ka_t))dk \right| \leq \epsilon.$$

Hence we look more precisely at the second integral. Using once again the uniform continuity, let U be a neighborhood of the identity such that for all $x \in U(S)$, $z' \in \mathbf{P}^1(\mathbf{C})$ and $u \in U$, we have:

$$|\beta(xu, z') - \beta(x, z')| \leq \epsilon.$$

Using the wavefront lemma, we have an open set $V \subset P$ such that $KVa_t \subset Ka_tU$ for all $t > 0$. That means that for any $k \in K$, $p \in V$ and $t > 0$, we have

$$|\beta(kpa_t, z) - \beta(ka_t, z)| \leq \epsilon.$$

Moreover, as the (right-)action of P stabilizes the positive endpoint of the geodesic, we have, for any $k \in K$ and $p \in V$:

$$z^+(kp) = z^+(k).$$

Hence, for each $p \in V$ we may compute (see also [EM93, pp. 16-17]:

$$\begin{aligned} \left| \int_{Kp} \beta(\sigma^+(ka_t))dk - \int_K \beta(\sigma^+(ka_t))dk \right| &\leq \int_K \left|\beta(kpa_t, z^+(k)) - \beta(kpa_t, z^+(k))\right| dk \\ &\leq \epsilon \end{aligned}$$

We may integrate this estimation over $p \in V$. Let χ_{KV} be the indicator function of KV. It belongs to $\mathrm{L}^2(X)$. And we have, by the previous computation:

$$\left| \frac{1}{\mathcal{L}(KV)} \int_{KV} \beta(\sigma^+(xa_t))dx - \int_K \beta(\sigma^+(ka_t))dk \right| \leq \epsilon.$$

Moreover, as a direct consequence of the mixing property (with $\alpha = \frac{\chi_{KV}}{\mathcal{L}(KV)}$), we get:

$$\frac{1}{\mathcal{L}(KV)} \int_{KV} \beta(\sigma^+(xa_t))dx \xrightarrow{t\to\infty} \frac{\int_X \beta \circ \sigma^+}{\mathcal{L}(X)}.$$

Eventually, by definition of μ^+, we have the equality $\int_X \beta \circ \sigma^+ = \int_{U(M_\rho)} \beta d\mu^+$. □

References

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