The Local Langlands correspondence: from extended quotients to affine Hecke algebras

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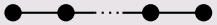
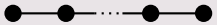
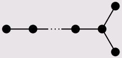
The main actors

- G group of F-rational points of a connected reductive algebraic F-group, with F a non-archimedean local field (finite extension of $\mathbb{Q}_p$ or of $\mathbb{F}_p((t))$). We will refer to G as a *p*-adic group.
- $G^\vee$ complex reductive group with root datum dual to that of G

Examples:

G	Dynkin diagram	$G^\vee$
$\mathrm{GL}_n(\mathbb{Q}_p)$		$\mathrm{GL}_n(\mathbb{C})$
$\mathrm{SL}_n(\mathbb{Q}_p)$		$\mathrm{PSL}_n(\mathbb{C})$
$\mathrm{PGL}_n(\mathbb{Q}_p)$		$\mathrm{SL}_n(\mathbb{C})$
$\mathrm{Sp}_{2n}(\mathbb{Q}_p)$		$\mathrm{SO}_{2n+1}(\mathbb{C})$
$\mathrm{SO}_{2n+1}(\mathbb{Q}_p)$		$\mathrm{Sp}_{2n}(\mathbb{C})$
$\mathrm{SO}_{2n}(\mathbb{Q}_p)$		$\mathrm{SO}_{2n}(\mathbb{C})$
$\mathrm{G}_2(\mathbb{Q}_p)$		$\mathrm{G}_2(\mathbb{C})$

The Local Langlands Correspondence (LLC)

predicts a surjective map, satisfying several properties,

$$\left\{ \begin{matrix} \text{irred. smooth} \\ \text{repres. } \pi \text{ of } G \end{matrix} \right\} / \text{iso.} \xrightarrow{\mathcal{L}} \left\{ \begin{matrix} L\text{-parameters} \\ \text{i.e. cont. homomorphisms} \\ \varphi_\pi \colon W_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G \end{matrix} \right\} / G^\vee\text{-conj.},$$

where $W_F :=$ absolute Weil group of F and ${}^L G := G^\vee \rtimes W_F$.
The fibers of $\mathcal{L}$, called L-packets, are expected to be finite.

Remark

In order to obtain a bijection LLC between the group side and the Galois side, the map $\mathcal{L}$ was later enhanced: on the Galois side, one considers enhanced L-parameters: (φ_π, ρ_π), where the enhancement ρ_π is a representation of a certain component group attached to φ_π.
$\Phi_{\mathrm{e}}(G) :=$ set of $G^\vee$-conjugacy classes of enhanced L-paramters.

A bijective LLC has been constructed in particular in the following cases:

- $G = F^\times$ Class field theory (first half of the 20th century);
- $G = \mathrm{GL}_n(F)$ Harris-Taylor (1998), Henniart (2000), Scholze (2010);
- $G = \mathrm{SL}_n(F)$ (and its inner twists) Hiraga-Saito $\mathrm{char}(F) = 0$ (2012); A.-Baum-Plymen-Solleveld $\mathrm{char}(F) > 0$ (2016).
- $G = \mathrm{Sp}_{2n}(F), \mathrm{SO}_{2n+1}(F)$ $(\mathrm{char}(F) = 0)$ Arthur (2013);
- $G = \mathrm{G}_2(F)$ A.-Xu (2022), Gan-Savin (2022).

Spectral extended quotient

Let X be a space and Γ a finite group acting on X. For $x \in X$, let $\Gamma_x \subset \Gamma$ be the fixator of x: $\Gamma_x := \{\gamma \in \Gamma \,:\, \gamma \cdot x = x\}$.
The (spectral) extended quotient of X by Γ is the quotient

$$X/\!/\Gamma := \{(x, \tau) \,:\, x \in X, \tau \in \mathrm{Irr}(\Gamma_x)\} / \Gamma.$$

Example 1

Let T be a torus in a reductive group G and $W := \mathrm{N}_G(T)/T$ the corresponding Weyl group, acting on T by conjugation, then we can consider the extended quotient $T//W$.

Example 2

G a p-adic group, L Levi subgroup of G and $\mathrm{Irr}_{\mathrm{cusp}}(L)$ set of isomorphism classes of supercuspidal irrep. of L.
The group $W(L) := \mathrm{N}_G(L)/L$ acts on $\mathrm{Irr}_{\mathrm{cusp}}(L)$ and we can form the panoramic (spectral) p-adic extended quotient:

$$\mathrm{Irr}_{\mathrm{cusp}}(L)//W(L).$$

Example 3

$G^\vee$ and $L^\vee$ complex dual groups of G, L and $\Phi_{\mathrm{e,cusp}}(L)$ the set of $L^\vee$-conjugacy classes of cuspidal enhanced Langlands parameters for L. The group $W(L^\vee) := \mathrm{N}_{G^\vee}(L^\vee)/L^\vee$ acts on $\Phi_{\mathrm{e,cusp}}(L)$ and we can form the panoramic (spectral) Galois extended quotient:

$$\Phi_{\mathrm{e,cusp}}(L)/\!/W(L^\vee).$$

Remark

The groups $W(L)$ and $W(L^\vee)$ are canonically isomorphic.

Conjecture

The local Langlands correspondence induces a bijection

$$\mathrm{Irr}_{\mathrm{cusp}}(L)/\!/W(L) \xleftrightarrow{1-1} \Phi_{\mathrm{e,cusp}}(L)/\!/W(L^\vee)$$

for any Levi subgroup L of G.

Notation

- $\mathcal{G}$ complex (possibly disconnected) reductive group
- $\mathcal{P}$ parabolic subgroup of $\mathcal{G}$ (i.e., subgroup of $\mathcal{G}$ s.t. $\mathcal{P}^\circ$ is a parabolic subgroup of $\mathcal{G}^\circ$)
- $\mathcal{L}$ complement in $\mathcal{P}$ of its unipotent radical $\mathcal{U}$
- $\mathrm{Unip}_{\mathcal{G}}$ unipotent variety of $\mathcal{G}$, similarly, $\mathrm{Unip}_{\mathcal{P}}$, $\mathrm{Unip}_{\mathcal{L}}$
- $D_c^b(X)$ category of bounded constructible ℓ-adic sheaves on the algebraic stack X.

Geometric parabolic induction

We consider the correspondence of algebraic stacks

$$\mathrm{Unip}_{\mathcal{L}}/\mathcal{L} \xleftarrow{\pi} \mathrm{Unip}_{\mathcal{P}}/\mathcal{P} \xrightarrow{\iota} \mathrm{Unip}_{\mathcal{G}}/\mathcal{G}$$

induced by the natural maps $\pi\colon \mathcal{P} \twoheadrightarrow \mathcal{L}$ and $\mathcal{P} \hookrightarrow \mathcal{G}$.
The functor $\mathrm{i}_{\mathcal{L},\mathcal{P}}^{\mathcal{G}}\colon D_c^b(\mathrm{Unip}_{\mathcal{L}}/\mathcal{L}) \to D_c^b(\mathrm{Unip}_{\mathcal{G}}/\mathcal{G})$ is defined by

$$\mathrm{i}_{\mathcal{L},\mathcal{P}}^{\mathcal{G}} := \iota_! \circ \pi^*.$$

Definition of cuspidality

Let $\mathcal{E}$ be an irreducible $\mathcal{L}$-equivariant local system on a unipotent class C in $\mathcal{L}$. We say that the pair $(C, \mathcal{E})$ is cuspidal if the intersection cohomology sheaf $\mathrm{IC}(C, \mathcal{E})$ does not occur in $\mathrm{i}^{\mathcal{G}}_{\mathcal{L},\mathcal{P}}(D^b_c(\mathrm{Unip}_{\mathcal{L}}/\mathcal{L}))$ for any proper Levi subgroup $\mathcal{L}$ of $\mathcal{G}$.

Strategy

We will plug the above construction into the Galois side of the correspondence.

Notation

We set $W_F' := W_F \times \mathrm{SL}_2(\mathbb{C})$, and define

$$S_\varphi := \mathrm{Z}_{G^\vee}(\varphi(W_F')) \tag{1}$$

if G is a pure inner twist of a quasi-split group. (There are variants for other cases.)

Definition

An enhanced L-parameter is a pair (φ, ρ) where φ is an L-parameter for G and $\rho \in \mathrm{Irr}(\mathcal{S}_\varphi)$, with $\mathcal{S}_\varphi := \pi_0(S_\varphi) = S_\varphi / S_\varphi^\circ$.

We set $\mathcal{G} = \mathcal{G}_\varphi := \mathrm{Z}_{G^\vee}(\varphi(W_F))$. We have

$$\mathcal{S}_\varphi \simeq \pi_0(\mathrm{Z}_{\mathcal{G}_\varphi}(u)), \quad \text{where } u = u_\varphi := \varphi(1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)).$$

Definition

An enhanced L-parameter $(\varphi, \rho) \in \Phi_{\mathrm{e}}$ is called cuspidal if the following properties hold:

- φ is discrete (i.e., $\varphi(W_F \times \mathrm{SL}_2(\mathbb{C}))$ is not contained in any proper Levi subgroup of $G^\vee$),
- (u_φ, ρ) is a *cuspidal pair* in $\mathcal{G}_\varphi$.

The cuspidality conjecture (special case of the above for $L = G$)

The local Langlands correspondence restricts to a bijection

$$\mathrm{Irr}_{\mathrm{cusp}}(G) \xleftrightarrow{1-1} \Phi_{\mathrm{e,cusp}}(G).$$

State of art

The cuspidality conjecture is known to hold for all the Levi subgroups (including the groups themselves) of

- general linear groups and split classical p-adic groups [Moussaoui, 2017],
- inner forms of linear groups and of special linear groups, and quasi-split unitary p-adic groups [A-Moussaoui-Solleveld, 2018],
- the p-adic group G_2 [A-Xu, 2022],
- pure inner forms of quasi-split classical p-adic groups [A-Moussaoui-Solleveld, 2022].

Property $C(L)$

There is a bijection $\mathfrak{L}_L \colon \mathrm{Irr}_{\mathrm{cusp}}(L) \xrightarrow{1-1} \Phi_{\mathrm{e,cusp}}(L)$ such that

$$\mathfrak{L}_L \circ \mathrm{Ad}(w) = \mathrm{Ad}(w^\vee) \circ \mathfrak{L}_L \quad \text{for any } w \in W_G(L)$$

where $w \mapsto w^\vee$ denotes the canonical bijection $W(L) \to W(L^\vee)$.

Remark

(Cuspidality conjecture) $\Leftrightarrow$ (Property $C(G)$ is satisfied with $\mathfrak{L}_G = \mathrm{LLC}$).

A Langlands correspondence between panoramic extended quotients:

If Property $C(L)$ is satisfied for L a Levi subgroup of G, then $\mathfrak{L}_L$ induces a bijection

$$\mathrm{Irr}_{\mathrm{cusp}}(L)/\!/W_G(L) \xleftrightarrow{1-1} \Phi_{\mathrm{e,cusp}}(L)/\!/W_{G^\vee}(L^\vee). \tag{2}$$

Theorem [A-Moussaoui-Solleveld, 2018]

For any G, there is a bijection:

$$\Phi_{\mathrm{e}}(G) \xleftrightarrow{1-1} \bigsqcup_{L \in \mathfrak{L}(G)} (\Phi_{\mathrm{e,cusp}}(L)//W_{G^\vee}(L^\vee))_{L\natural}.$$

Theorem [Solleveld, 2020]

For any G, there is a bijection:

$$\mathrm{Irr}(G) \xleftrightarrow{1-1} \bigsqcup_{L \in \mathfrak{L}(G)} (\mathrm{Irr}_{\mathrm{cusp}}(L)//W_G(L))_{\natural}.$$

Consequence

If Property $C(L)$ is satisfied for all the Levi subgroups L of G (including $L = G$), and if the bijection (2) is "compatible with the twists", then we get a bijection (hopefully the LLC)

$$\mathrm{Irr}(G) \xleftrightarrow{1-1} \Phi_{\mathrm{e}}(G).$$

Notation

- $\mathfrak{X}_{\mathrm{nr}}(L)$ group of unramified characters of L (a character is unramified if it is trivial on every compact subgroup of L).
- $\mathfrak{s} = \mathfrak{s}_G := [L, \sigma]_G$ the G-conjugacy class of the pair $(L, \mathfrak{X}_{\mathrm{nr}}(L) \cdot \sigma)$, where $\sigma \in \mathrm{Irr}_{\mathrm{cusp}}(L)$
- $\mathrm{Irr}^{\mathfrak{s}}(G)$: set of (isomorphism classes of) irreducible representations of G whose supercuspidal support lies in $\mathfrak{s}$
- $\mathfrak{X}_{\mathrm{nr}}(L, \sigma) := \{\chi \in \mathfrak{X}_{\mathrm{nr}}(L) \, : \, \sigma \otimes \chi \cong \sigma\}$
- the bijection $\mathfrak{X}_{\mathrm{nr}}(L)/\mathfrak{X}_{\mathrm{nr}}(L, \sigma) \to \mathrm{Irr}^{\mathfrak{s}_L}(L)$, $\chi \mapsto \sigma \otimes \chi$ endows $\mathrm{Irr}^{\mathfrak{s}_L}(L)$ with the structure of a complex torus $T^{\mathfrak{s}}$
- $W^{\mathfrak{s}} := \mathrm{N}_G(\mathfrak{s})/L$ acts on $T^{\mathfrak{s}}$ by automorphisms of algebraic varieties
- $\mathfrak{B}(G)$ set of such classes $\mathfrak{s}$.

The partition of $\mathrm{Irr}(G)$ into Bernstein series

We have [Bernstein, 1984]:

$$\mathrm{Irr}(G) = \bigsqcup_{\mathfrak{s}\in\mathfrak{B}(G)} \mathrm{Irr}^{\mathfrak{s}}(G). \tag{3}$$

Moreover, for every $\mathfrak{s} \in \mathfrak{B}(G)$ [Solleveld, 2020]:

$$\mathrm{Irr}^{\mathfrak{s}}(G) \xleftrightarrow{1-1} (T^{\mathfrak{s}}/W^{\mathfrak{s}})_{\natural}. \tag{4}$$

Example

Suppose G is F-split and let $\mathfrak{s}^1 := [T, \mathrm{triv}]_G$, with T an F-split maximal torus. We have

$$\mathrm{Irr}^{\mathfrak{s}^1}(G) = \{\text{Iwahori-spherical irreps. of } G\} / \sim .$$

Slogan: There is a similar decomposition on the Galois side.

Notation

- $L^\vee$ Langlands dual group of L
- If G is F-split: $\mathfrak{X}_{\mathrm{nr}}({}^L L) := \{\zeta\colon W_F/I_F \to \mathrm{Z}^\circ_{L^\vee}\}$, where I_F is the inertia group of F. In general, $\mathfrak{X}_{\mathrm{nr}}({}^L L) :=$
- $\mathfrak{s}^\vee = \mathfrak{s}^\vee_{G^\vee} := [L^\vee \rtimes W_F, (\varphi_0, \rho_0)]_{G^\vee}$ the $G^\vee$-conjugacy class of $(L^\vee \rtimes W_F, \mathfrak{X}_{\mathrm{nr}}({}^L L)\cdot(\varphi_0, \rho_0))$, where $(\varphi_0, \rho_0) \in \Phi_{\mathrm{e,cusp}}(L)$
- $\Phi_{\mathrm{e}}^{\mathfrak{s}^\vee}(G)$ enhanced L-parameters whose cuspidal support lies in $\mathfrak{s}^\vee$
- $W^{\mathfrak{s}^\vee} := \mathrm{N}_{G^\vee}(\mathfrak{s}^\vee)/L^\vee$ acts on $\Phi_{\mathrm{e}}^{\mathfrak{s}^\vee_{L^\vee}}(L)$
- $\mathfrak{B}^\vee(G)$ the set of such $\mathfrak{s}^\vee$.

Theorem [A-Moussaoui-Solleveld, 2018]

The set $\Phi_{\mathrm{e}}(G)$ is partitioned into series à la Bernstein as

$$\Phi_{\mathrm{e}}(G) = \bigsqcup_{\mathfrak{s}^\vee \in \mathfrak{B}(G^\vee)} \Phi_{\mathrm{e}}^{\mathfrak{s}^\vee}(G). \tag{5}$$

Moreover, for every $\mathfrak{s}^\vee \in \mathfrak{B}(G^\vee)$, we have

$$\Phi_{\mathrm{e}}^{\mathfrak{s}^\vee}(G) \xleftrightarrow{1-1} (\Phi_{\mathrm{e}}^{\mathfrak{s}_{L^\vee}^\vee}(L)//W^{\mathfrak{s}^\vee})_{L^\natural}. \tag{6}$$

Theorem

If G is

- an inner form of $\mathrm{GL}_n(F)$ [A-Baum-Plymen-Solleveld, 2019],
- the exceptional group of type G_2 [A-Xu, 2022],
- a pure inner form of a quasi-split classical p-adic group [A-Moussaoui-Solleveld, 2022],

then, for every $\mathfrak{s} = [L, \sigma]_G \in \mathfrak{B}(G)$ and for $\mathfrak{s}^\vee := [L^\vee \rtimes W_F, \mathrm{LLC}(\sigma)]_{G^\vee}$,

$$\mathrm{Irr}^{\mathfrak{s}}(G) \xrightarrow{1-1} \mathrm{Irr}^{\mathfrak{s}_L}(L)/\!/W^{\mathfrak{s}} \xrightarrow{1-1} \Phi_{\mathrm{e}}^{\mathfrak{s}_{L^\vee}^\vee}(L)/\!/W^{\mathfrak{s}^\vee} \xrightarrow{1-1} \Phi_{\mathrm{e}}^{\mathfrak{s}^\vee}(G)$$

coincides with the LLC, and the following diagram is commutative

$$\begin{array}{ccc}
\mathrm{Irr}^{\mathfrak{s}}(G) & \xrightarrow[1-1]{\mathrm{LLC}} & \Phi_{\mathrm{e}}^{\mathfrak{s}^\vee}(G) \\
\Big\downarrow{\scriptstyle \mathrm{Sc}} & & \Big\downarrow{\scriptstyle \mathrm{Sc}} \\
\mathrm{Irr}^{\mathfrak{s}_L}(L) & \xrightarrow[\mathrm{LLC}]{1-1} & \Phi_{\mathrm{e}}^{\mathfrak{s}_{L^\vee}^\vee}(L)
\end{array}$$

Theorem [A-Moussaoui-Solleveld, 2018]

There exist a (twisted) extended affine Hecke algebra $\mathcal{H}(G^\vee, \mathfrak{s}^\vee)$ such that

$$\Phi_{\mathrm{e}}^{\mathfrak{s}^\vee}(G) \overset{1-1}{\longleftrightarrow} \mathrm{Irr}(\mathcal{H}(G^\vee, \mathfrak{s}^\vee)).$$

The Bernstein decomposition [Bernstein, 1984]

The category $\mathfrak{R}(G)$ of smooth representations of a p-adic group G is a direct product

$$\mathfrak{R}(G) = \prod_{\mathfrak{s} \in \mathfrak{B}(G)} \mathfrak{R}^{\mathfrak{s}}(G) \tag{7}$$

of the full subcategories $\mathfrak{R}^{\mathfrak{s}}(G)$, where $\mathfrak{R}^{\mathfrak{s}}(G)$ is the subcategory of $\mathfrak{R}(G)$ whose objects are the representations π such that every irreducible G-subquotient of π has its supercuspidal support in $\mathfrak{s}$.

Theorem ([A.-Moussaoui-Solleveld], [A.-Xu])

If G an inner form of $\mathrm{GL}_n(F)$, the group G_2, or a pure inner form of a quasi-split classical p-adic group, then

$$\mathfrak{R}^{\mathfrak{s}}(G) \overset{\text{Morita}}{\sim} \mathrm{Mod}(\mathcal{H}(G,\mathfrak{s})),$$

where $\mathcal{H}(G,\mathfrak{s})$ is an affine Hecke algebra, which is isomorphic to $\mathcal{H}(G^\vee,\mathfrak{s}^\vee)$.

The Bernstein decomposition of the Hecke algebra of G

Let $\mathcal{H}(G)$ be the convolution algebra of locally constant, compactly supported functions $f\colon G \to \mathbb{C}$. By letting G act on $\mathcal{H}(G)$ by left translation, we obtain a decomposition

$$\mathcal{H}(G) = \bigoplus_{\mathfrak{s}\in\mathfrak{B}(G)} \mathcal{H}(G)^{\mathfrak{s}}, \tag{8}$$

with $\mathcal{H}(G)^{\mathfrak{s}} \in \mathfrak{R}^{\mathfrak{s}}(G)$. The spaces $\mathcal{H}(G)^{\mathfrak{s}}$ are two-sided ideals of $\mathcal{H}(G)$.

A version with C^*-algebras

- From the point of view of noncommutative geometry, for the study of tempered representations of G, it is interesting to use the reduced C^*-algebra $C^*_{\mathrm{r}}(G)$ of G (i.e. the completion of $\mathcal{H}(G)$ in the algebra of bounded linear operators on the Hilbert space $L^2(G)$).
- The spectrum of $C^*_{\mathrm{r}}(G)$ coincides with the tempered dual $\mathrm{Irr}^{\mathrm{temp}}(G)$ of G.
- We have the following decomposition of $C^*_{\mathrm{r}}(G)$:

$$C^*_{\mathrm{r}}(G) = \bigoplus_{\mathfrak{s} \in \mathfrak{B}(G)} C^*_{\mathrm{r}}(G)^{\mathfrak{s}}.$$

Conjecture (topological K-theory version of the ABPS Conjecture)

Let $\mathfrak{s} \in \mathfrak{B}(G)$ and let $T_{\mathrm{un}}^{\mathfrak{s}}$ be the set of unitary representations in $T^{\mathfrak{s}}$, a $W^{\mathfrak{s}}$-stable compact real subtorus. There exists a canonical isomorphism

$$K_{W^{\mathfrak{s}}}^{*}(T_{\mathrm{un}}^{\mathfrak{s}}) \to K(C_{\mathrm{r}}^{*}(G)^{\mathfrak{s}})$$

where $K_{W^{\mathfrak{s}}}^{j}(T_{\mathrm{un}}^{\mathfrak{s}})$ is the classical topological equivariant K-theory for the group $W^{\mathfrak{s}}$ acting on $T_{\mathrm{un}}^{\mathfrak{s}}$.

Remark

If G is a classical group, then $C_{\mathrm{r}}^{*}(G)^{\mathfrak{s}}$ is Morita equivalent to the reduced C^{*}-completion of $\mathcal{H}(G, \mathfrak{s})$ (i.e. the closure of $\mathcal{H}(G, \mathfrak{s})$ in the algebra of bounded linear operators on the Hilbert space completion of $\mathcal{H}(G, \mathfrak{s})$).

Theorem [A., 2023]

Let $(G, G') = (\mathrm{Sp}_{2n}(F), \mathrm{O}_{2n'}(F))$, viewed as a dual pair in $\mathrm{Sp}_{2nn'}(F)$, and fix $\mathfrak{s} \in \mathfrak{B}(G)$.

1. The images by the Howe correspondence of all the $\pi \in \mathrm{Irr}^{\mathfrak{s}}(G)$ belong to a unique Bernstein series $\mathrm{Irr}^{\theta(\mathfrak{s})}(G')$ of G'.
2. The Howe correspondence for (G, G') induces a correspondence between simple modules of the extended affine Hecke algebras $\mathcal{H}(G_n, \mathfrak{s})$ and $\mathcal{H}(G'_m, \theta(\mathfrak{s}))$, and hence a correspondence $\theta^{\mathfrak{s}}$ between $\Phi_{\mathrm{e}}^{\mathfrak{s}}(G)$ and $\Phi_{\mathrm{e}}^{\theta(\mathfrak{s})}(G')$.
3. When $n' = n$ or $n' = n + 1$, the Howe correspondence induces a correspondence between simple modules of $C_{\mathrm{r}}^*(G)^{\mathfrak{s}}$ and $C_{\mathrm{r}}^*(G)^{\theta(\mathfrak{s})}$.

Thank you very much for your attention!